Journal of Solid Mechanics Vol. 6, No. 1 (2014) pp. 28-42

Vibration Analysis for Rectangular Plate Having a Circular Central Hole with Point Support by Rayleigh-Ritz Method

K. Torabi*, A.R. Azadi

Department of Mechanical Engineering, University of Kashan, Kashan, Iran

Received 2 September 2013; accepted 8 December 2013

ABSTRACT

In this paper, the transverse vibrations of rectangular plate with circular central hole have been investigated and the natural frequencies of the mentioned plate with point supported by Rayleigh-Ritz Method have been obtained. In this research, the effect of the hole is taken into account by subtracting the energies of the hole domain from the total energies of the whole plate. To determine the kinetic and potential energies of plate, admissible functions for rectangular plate are considered as beam functions and it has been tried that the functions of the deflection of plate, in the form of polynomial functions proportionate with finite degrees, to be replaced by Bessel function, which is used in the analysis of the vibrations of a circular plate. Consideration for a variety of edge conditions is given through a combination of simply supported, clamped and free boundary conditions. In this study, the effects of increasing the diameter of the hole and the effects of number of point supported on the natural frequencies were investigated and the optimum radius of the circular hole for different boundary conditions are obtained. The method has been verified with many known solutions. Furthermore, the convergence is very fast with any desirable accuracy to exact known natural frequencies.

Keywords: Rectangular plate; Circular plate; Rayleigh-Ritz method; Hole; Vibration; Point support

1 INTRODUCTION

RECTANGULAR plate with a rectangular or a circular hole has been widely used as a substructure for ship, airplane, and plant. Uniform circular, annular and rectangular plates have been also widely used as structural components for various industrial applications and their dynamic behaviors can be described by exact solutions. However, the vibration characteristics of a rectangular plate with an eccentric circular hole can not be analyzed easily. Perforated plates or plates with cut-outs are commonly encountered in engineering practice. Cut-outs are introduced to provide access, reduce weight, and alter the dynamic response of structures. Furthermore, structure-borne noise generated by machinery such as the diesel engines, gearboxes, generators, and auxiliary machinery are also radiated by these plate structures and should be suppressed in the various operating conditions.

Rectangular plates with point supports can model several structures of practical interest, such as slabs supported on columns, printed circuit boards or solar panels supported at a few points.

The vibration characteristics of a rectangular plate with a hole can be solved by either the Rayleigh-Ritz method or the finite element method. The Rayleigh-Ritz method is an effective method when the rectangular plate has a rectangular hole. However, it cannot be easily applied to the case of a rectangular plate with a circular hole since the

* Corresponding author. Tel.: +98 361 591 2448; Fax: +98 361 555 9930.
E-mail address: kvntrb@kashanu.ac.ir (K. Torabi).

admissible functions for the rectangular hole domain do not permit closed-form integrals. Many studies have been done on the subject, some of which are mentioned in this section.

Monahan et al. [1] applied the finite element method to a clamped rectangular plate with a rectangular hole and verified the numerical results by experiments. Paramasivam [2] used the finite difference method for a simply-supported and clamped rectangular plate with a rectangular hole. There are many research works concerning plate with a single hole but a few works on plate with multiple holes. Aksu and Ali [3] also used the finite difference method to analyze a rectangular plate with more than two holes. Rajamani and Prabhakaran [4] assumed that the effect of a hole is equivalent to an externally applied loading and carried out a numerical analysis based on this assumption for a composite plate. Rajamani and Prabhakaran [5] investigated the effect of a hole on the natural vibration characteristics of isotropic and orthotropic plates with simply-supported and clamped boundary conditions. Ali and Atwal [6] applied the Rayleigh-Ritz method to a simply-supported rectangular plate with a rectangular hole, using the static deflection curves for a uniform loading as admissible functions. Lam et al. [7] divided the rectangular plate with a hole into several sub areas and applied the modified Rayleigh-Ritz method. Lam and Hung [8] applied the same method to a stiffened plate. Laura et al. [9] calculated the natural vibration characteristics of a simply-supported rectangular plate with a rectangular hole by the classical Rayleigh-Ritz method. Sakiyama et al. [10] analyzed the natural vibration characteristics of an orthotropic plate with a square hole by means of the Green function assuming the hole as an extremely thin plate. The vibration analysis of a rectangular plate with a circular hole does not lend an easy approach since the geometry of the hole is not the same as the geometry of the rectangular Plate. Takahashi (1958) used the classical Rayleigh-Ritz method after deriving the total energy by subtracting the energy of the hole from the energy of the whole plate. He employed the eigenfunctions of a uniform beam as admissible functions. Joga-Rao and Pickett [11] proposed the use of algebraic polynomial functions and biharmonic singular functions. Kumai [12] Hegarty [13], Eastep and Hemmig [14], and Nagaya [15-16] used the point-matching method for the analysis of a rectangular plate with a circular hole. The point-matching method employed the polar coordinate system based on the circular hole and the boundary conditions were satisfied along the points located on the sides of the rectangular plate. Lee and Kim [17] carried out vibration experiments on the rectangular plates with a hole in air and water. Kim et al. [18] performed the theoretical analysis on a stiffened rectangular plate with a hole. Avalos and Laura [19] calculated the natural frequency of a simply-supported rectangular plate with two rectangular holes using the Classical Rayleigh-Ritz method. Lee et al. [20] analyzed a square plate with two collinear circular holes using the classical Rayleigh-Ritz method. A circular plate with en eccentric circular hole has been treated by various methods. Khurasia and Rawtani [21] studied the effect of the eccentricity of the hole on the vibration characteristics of the circular plate by using the triangular finite element method. Lin [22] used an analytical method based on the transformation of Bessel Functions to calculate the free transverse vibrations of uniform circular plates and membranes with eccentric holes. Laura et al. [23] applied the Rayleigh-Ritz method to circular plates restrained against rotation with an eccentric circular perforation with a free edge. Cheng et al. [24] used the finite element analysis code, Nastran, to analyze the effects of the hole eccentricity, hole size and boundary condition on the vibration modes of annular-like plates. Lee et al. [25] used an indirect formulation in conjunction with degenerate kernels and Fourier series to solve for the natural frequencies and modes of circular plates with multiple circular holes and verified the finite element solution by using ABAQUS. Zhong and Yu. [26] Formulated a weak-form quadrature element method to study the flexural vibrations of an eccentric annular Mindlin plate. Wang [27], the Ritz method is used to determine the minimum stiffness location of the elastic point support for raising the fundamental natural frequency of a rectangular plate to the second frequency of the unsupported plate, which usually is the upper limit of the first frequency for a single support. Joseph Watkins et all. [28] Studied the vibration of an elastically point supported rectangular plate using eigensensitivity analysis. Lorenzo [29] is employed the trigonometric functions as admissible solutions in the Ritz method for general vibration analysis of rectangular orthotropic Kirchhoff plates.

As it was mentioned earlier, in most of the researches done in this field, Rayleigh-Ritz method and numerical methods have been used and with the help of reducing the hole energy comparing to the energy of the whole rectangular plate, the problem has been analyzed. Also for studying the issues considering the position conditions and angle, the quantity of so many points in the edges of the rectangular plate have been used. In this study, the analysis of transverse vibrations of rectangular plate with circular central hole with different point support is studied and the natural frequencies and natural modes of a rectangular plate with circular hole have been obtained. In this method, a simple polynomial functions, the desired frequency range, which can replace the Bessel functions will be used, and convergence the problem will be obtained easily.

2 FORMULATION OF THE PROBLEM

2.1 Applying the Rayleigh-Ritz approach to rectangular plate

From the vibration theory of thick plates, the strain energy U_p and kinetic energy T_p of an elastic isotropic rectangular plate in the cartesian coordinate can be written as follows:

$$U_p = \frac{1}{2} D \int_0^a \int_0^b \left[\left(\frac{\partial^2 W}{\partial x^2} \right)^2 + \left(\frac{\partial^2 W}{\partial y^2} \right)^2 + 2\nu \left(\frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} \right) + 2(1-\nu) \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 \right] dx\, dy \tag{1a}$$

$$T_p = \frac{1}{2} \rho h \int_0^a \int_0^b \dot{W}^2 \, dx\, dy \tag{1b}$$

where ρ is the mass density of the material, and D is the plate flexural rigidity defined as:

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{2}$$

Here, E is Young's modulus and ν is Poisson's ratio. With side lengths a in the X direction and b in the Y, Taking the following non-dimensional coordinates:

$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{b}, \quad \alpha = \frac{a}{b}. \tag{3}$$

The Ritz approximation is employed by assuming the following solution:

$$W(\xi,\eta,t) = \varphi(\xi,\eta) Q(t) \tag{4}$$

where $\varphi(\xi,\eta) = \left[\varphi_1 \ \ \varphi_2 \ ..... \ \varphi_m \right]$ is a $1 \times m$ matrix consisting of the admissible functions and $Q(t) = \left[Q_1 \ \ Q_2 \ ..... \ Q_m \right]$ is a $m \times 1$ vector consisting of generalized coordinates, in which m is the number of admissible functions used for the approximation of the deflection.

$$\varphi(\xi,\eta) = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} \phi_m(\xi) \psi_n(\eta) \tag{5}$$

In which $\phi_m(\xi)$ and $\psi_n(\eta)$ denote the assumed admissible functions in the $x-$and $y-$directions, respectively, with substituting Eq (4) into Eq (1) results in Eq (6) .

$$T_p = \frac{1}{2} \dot{Q}^T M_p \, \dot{Q}, \qquad U_p = \frac{1}{2} Q^T K_p \, Q \tag{6}$$

where

$$M_p = \rho h a b \bar{M}_p, \qquad K_p = \frac{Db}{a^3} \bar{K}_p, \tag{7}$$

In which case

$$\bar{M}_p = \int_0^1 \int_0^1 \ \varphi^T \varphi \, d\xi \, d\eta \tag{8a}$$

$$\bar{K}_p = \int_0^1\int_0^1 \left[\frac{\partial^2\varphi^T}{\partial\xi^2}\frac{\partial^2\varphi}{\partial\xi^2} + \alpha^4 \frac{\partial^2\varphi^T}{\partial\eta^2}\frac{\partial^2\varphi}{\partial\eta^2} + \nu\alpha^2 \left(\frac{\partial^2\varphi^T}{\partial\xi^2}\frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial^2\varphi^T}{\partial\eta^2}\frac{\partial^2\varphi}{\partial\xi^2} \right) + 2(1-\nu)\alpha^2 \frac{\partial^2\varphi^T}{\partial\xi\partial\eta}\frac{\partial^2\varphi}{\partial\xi\partial\eta} \right] d\xi\, d\eta \tag{8b}$$

$\bar{M}$ and $\bar{K}$ represent the non-dimensionalized mass and stiffness matrice. After substituting the plate displacement function in Eq (5) into the above energy expressions, a set of $m\times n$ homogeneous equations of A_{mn} is then formulated by differentiating the Lagrangian energy, defined by $U-T$, with regard to each of the undetermined coefficient A_{mn} . Afrer choosing a set of appropriate admissible function for ϕ and ψ , the eigenvalue equation can be derived as:

$$\left[\bar{K}_p - \lambda^2 \bar{M}_p\right]\{A\} = 0 \tag{9}$$

where $\lambda = \omega a^2 \sqrt{\dfrac{\rho h}{D}}$ is the natural frequency parameter. Then, the non-dimensionalized mass and stiffness matrices given by Eq (5) can be expressed as [28-30].

$$(\bar{M}_p)_{ij} = \int_0^1 \phi_i\phi_j d\xi \int_0^1 \psi_i\psi_j d\eta, \quad i, j = 1, 2, \ldots m \tag{10a}$$

$$(\bar{K}_p)_{ij} = \int_0^1 \phi_i''\phi_j'' d\xi \int_0^1 \psi_i\psi_j d\eta + +\nu\alpha^2 \left[\int_0^1 \phi_i\phi_j'' d\xi \int_0^1 \psi_i''\psi_j d\eta + \int_0^1 \phi_i''\phi_j d\xi \int_0^1 \psi_i\psi_j'' d\eta \right] + \alpha^4 \int_0^1 \phi_i\phi_j d\xi \int_0^1 \psi_i''\psi_j'' d\eta + 2(1-\nu)\alpha^2 \int_0^1 \phi_i'\phi_j' d\xi \int_0^1 \psi_i'\psi_j' d\eta \tag{10b}$$

Here , $\bar{M}$ and $\bar{K}$ are a diagonal matrices. In this section, by considering the following as admissible function for the plate simply supported on all side, the boundary matrices will be applied easily.

$$\phi_i(\xi) = \sqrt{2}\sin(i\pi\xi), \quad i = 1, 2, \ldots \quad n \qquad \text{for } x-\text{direction} \tag{11a}$$

and

$$\psi_j(\eta) = \sqrt{2}\sin(j\pi\eta), \quad j = 1, 2, \ldots \quad n \qquad \text{for } y-\text{direction} \tag{11a}$$

In the case of the clamped condition in the $x-$direction, the eigen function of a clamped–clamped uniform beam can be used:

$$\phi_i(\xi) = \cosh(\beta_i\xi) - \cos(\beta_i\xi) - \gamma_i(\sinh(\beta_i\xi) - \sin(\beta_i\xi)), \quad i = 1, 2, \ldots \quad n \tag{12}$$

where β_i is obtained by solving the equation of $\cosh(\beta_i)\cos(\beta_i) - 1 = 0$ and $\beta_i = 4.730,\ 7.853, \ldots$ and $\gamma_i = \dfrac{\cosh(\beta_i) - \cos(\beta_i)}{\sinh(\beta_i) - \sin(\beta_i)}$.

In a similar manner, expressions for a plate with free edges in the $x-$direction, the eigenfunction of a free–free uniform beam:

$$\phi_i(\xi) = \cosh(\beta_i \xi) + \cos(\beta_i \xi) - \gamma_i (\sinh(\beta_i \xi) + \sin(\beta_i \xi)), \quad i = 1, 2, ... \quad n \tag{13}$$

where β_i and γ_i are the same as the ones for the clamped–clamped beam. For the admissible functions in the $y-$direction, the same method can be applied. The frequency parameter, λ is obtained by solving the generalized eigenvalue problem defined by Eq (9).

$$\det\left(\left[\bar{K}_p - \lambda^2 \bar{M}_p\right]\right) = 0 \tag{14}$$

2.2 Applying the Rayleigh-Ritz approach to circular plate

To obtain the natural frequencies of rectangular plate with circular central hole, It is similar to the rectangular plate, the mass and stiffness matrices are determined. From the vibration theory of circular plates, the strain energy U_C and kinetic energy T_C of an isotropic uniform circular plate with radius R and thickness h can be expressed as follows:

$$T_c = \frac{1}{2}\rho h \int_0^{2\pi}\int_0^R \dot{W}\, rdr\, d\theta \tag{15a}$$

$$U_c = \frac{1}{2} D \int_0^R \int_0^{2\pi} \left[\left(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} \right)^2 - 2(1-\nu) \left[\left(\frac{\partial^2 W}{\partial r^2} \right) \left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} \right) - \left(\frac{1}{r}\frac{\partial^2 W}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} \right)^2 \right] \right] rdr\, d\theta \tag{15b}$$

The Ritz approximation is employed by assuming the following solution:

$$W(r,\theta,t) = \varphi(r,\theta) Q_c(t) \tag{16}$$

where $\varphi(r,\theta) = \left[\varphi_1 \;\; \varphi_2 \; ..... \; \varphi_m\right]$ is a $1\times m$ matrix consisting of the admissible functions and $Q_c(t) = \left[Q_{1c} \;\; Q_{2c} \; ..... \; Q_{mc}\right]$ is a $m\times 1$ vector consisting of generalized coordinates, in which m is the number of admissible functions used for the approximation of the deflection [30]. With substituting Eq (16) into Eq (15) results in Eq (17).

$$T_C = \frac{1}{2}\dot{Q}_c^T M_c\, \dot{Q}_c, \qquad U_C = \frac{1}{2} Q_c^T K_c\, Q_c \tag{17}$$

where

$$M_c = \rho h \pi R^2 \bar{M}_c, \qquad K_c = \frac{\pi D}{R^2}\bar{K}_c \tag{18}$$

In which case

$$(\bar{M}_c)_{ij} = \int_0^{2\pi}\int_0^R \varphi_i \varphi_j\; rdr\, d\theta, \tag{19a}$$

$$(\bar{K}_c)_{ij} = \int_0^{2\pi}\int_0^R \left[\frac{\partial^2\varphi_i}{\partial r^2}\frac{\partial^2\varphi_j}{\partial r^2} + \left(\frac{1}{r}\frac{\partial\varphi_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi_i}{\partial\theta^2}\right)\left(\frac{1}{r}\frac{\partial\varphi_j}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi_j}{\partial\theta^2}\right) \right.$$
$$+\frac{\nu}{r}\left(\frac{\partial\varphi_i}{\partial r}\frac{\partial^2\varphi_j}{\partial r^2} + \frac{\partial\varphi_j}{\partial r}\frac{\partial^2\varphi_i}{\partial r^2} + \frac{1}{r}\frac{\partial^2\varphi_i}{\partial\theta^2}\frac{\partial^2\varphi_j}{\partial r^2} + \frac{1}{r}\frac{\partial^2\varphi_j}{\partial\theta^2}\frac{\partial^2\varphi_i}{\partial r^2}\right) \tag{19b}$$
$$\left. +2(1-\nu)\left(\frac{1}{r}\frac{\partial^2\varphi_i}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial^2\varphi_i}{\partial\theta}\right)\left(\frac{1}{r}\frac{\partial^2\varphi_j}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial^2\varphi_j}{\partial\theta}\right)\right] rdr\,d\theta.$$

The origin of the polar coordinate system is at the center of the circular plate. The boundary conditions possess symmetry with respect to the diameter of the circular plate. The deflection function in terms of Bessel functions and trigonometric functions is written as [30, 31]:

$$\varphi(r,\theta) = \sum_{n=0}^{\infty}\left\{A_n J_n(\lambda_n \frac{r}{R}) + B_n Y_n(\lambda_n \frac{r}{R}) + C_n I_n(\lambda_n \frac{r}{R}) + D_n K_n(\lambda_n \frac{r}{R})\right\} f_n(\theta) \tag{20}$$

where the coefficients A_n, B_n, C_n and D_n are determined from the boundary conditions and J_n and I_n are the Bessel function and the modified Bessel function of the first kind, Y_n and K_n are Bessel function and the modified Bessel function of the second kind of order n, respectively. Since the circular hole is to be free of all applied stress, the boundary conditions to be satisfied along the edge of the hole at $r = R$ are:

$$M_r = 0, \quad Q_r - \frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta} = 0. \tag{21a}$$

where M_r is the bending moment normal to the hole, $M_{r\theta}$ is the twisting moment in the same plane, and the Q_r is the shear force acting at the edge of the hole. For instance, if the boundary of the plate is considered to be clamped at the radius R of the plate then the boundary terms for solution $\varphi(r,\theta)$ can be written as:

$$\varphi(r,\theta) = 0, \quad \frac{\partial\varphi(r,\theta)}{\partial r} = 0. \tag{21b}$$

Also, solution $\varphi(r,\theta)$ must be finite at all points within the plate. This makes constants B_n, D_n vanish since the Bessel functions of second kind Y_n and K_n become infinite at $r = 0$. As it has been shown in Eq (20), deflection of the intended plate, can be expressed in terms of Bessel functions of the first kind. Due to the properties of the Bessel functions and regardless of terms with high degrees in Eq (20) and also obtained frequencies with the use of Finite Element method, in this section it has been tried to acquire the natural frequencies and the mode shapes of the rectangular plate with a central hole, with the use of polynomial functions proportionate with finite degrees in the intended frequency limits instead of the mentioned Bessel functions. Bessel functions of the first kind, denoted as $J_n(R)$, are solutions of Bessel's differential equation that are finite at the origin ($R = 0$) for integer n, and diverge as R approaches zero for negative non-integer n. The solution type (e.g., integer or non-integer) and normalization of $J_n(R)$ are defined by its properties below. It is possible to define the function by its Taylor series expansion around $R = 0$.

$$J_n(R) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+n+1)}(\frac{1}{2}R)^{2m+n} = P_n(R) \tag{22}$$

where $\Gamma(Z)$ is the gamma function. The series expansion for $I_n(R)$ is thus similar to that for $J_n(R)$, but without the alternating $(-1)^m$ factor.

$$I_n(R)=\sum_{m=0}^{\infty}\frac{1}{m\,!\Gamma(m+n+1)}(\frac{1}{2}R)^{2m+n}=S_n(R) \tag{23}$$

The series expansion for $Y_n(R)$ and $K_n(R)$ using a series expansion of Bessel functions $J_n(R)$ and $I_n(R)$ will be obtained easily. Here, $P_n(R), Q_n(R), S_n(R), T_n(R)$ are the polynomials with a limited degree, will be sought in the form of series expansions and in the desired frequency range, will be replaced by the Bessel functions $J_n(R), Y_n(R), I_n(R), K_n(R)$, respectively. Other relationships, by substituting the polynomials functions and simplify the equations will be obtained.

$$\varphi(r,\theta)=\sum_{n=0}^{\infty}\left\{A_nP_n(\lambda_nR)+B_nQ_n(\lambda_nR)+C_nS_n(\lambda_nR)+D_nT_n(\lambda_nR)\right\}f_n(\theta) \tag{24}$$

For example, if $n=1$, the following proposed polynomials, can be replaced by the Bessel functions.

$$\begin{aligned}
J_1&=P_1(R)=1-\frac{1}{4}\lambda_1^{\,2}R^2+\frac{1}{64}\lambda_1^{\,4}R^4-\frac{1}{2304}\lambda_1^{\,6}R^6,\\
Y_1&=Q_1(R)=\frac{2\log(\lambda_1R/2)}{\pi}+\frac{2\Gamma}{\pi}+\left(\frac{2-2\Gamma}{4\pi}\lambda_1^{\,2}-\frac{\log(\lambda R/2)}{2\pi}\lambda_1^{\,2}\right)R^2,\\
I_1&=S_1(R)=1+\frac{1}{4}\lambda_1^{\,2}R^2+\frac{1}{64}\lambda_1^{\,4}R^4+\frac{1}{2304}\lambda_1^{\,6}R^6,\\
K_1&=T_1(R)=-\log(\lambda_1R/2)-\Gamma+\left(\frac{2-2\Gamma}{8}\lambda_1^{\,2}-\frac{\log(\lambda_1R/2)}{4}\lambda_1^{\,2}\right)R^2+\left(\frac{3-2\Gamma}{128}\lambda_1^{\,4}-\frac{\log(\lambda_1R/2)}{64}\lambda_1^{\,4}\right)R^4,
\end{aligned} \tag{25}$$

After choosing a set of appropriate admissible function for φ, the eigenvalue equation for circular plate can be derived as:

$$\left[\bar{K}_c-\lambda^2\bar{M}_c\right]\left\{A\right\}=0 \tag{26}$$

And the frequency parameter for circular plate, λ is obtained by solving the generalized eigenvalue problem defined by Eq (27).

$$\det\left(\left[\bar{K}_c-\lambda^2\bar{M}_c\right]\right)=0 \tag{27}$$

where $\lambda^4=\dfrac{\rho hR^4\omega^2}{D}$.

2.3 Applying the Rayleigh-Ritz approach for rectangular plate with circular hole

For a rectangular plate with cutouts, the normal Rayleigh-Ritz method will require a beam function that is continuous over the plate domain while satisfying the inner and external boundary requirements. No such function has been reported in the open literature, and the analysis of such problem using the Rayleigh-Ritz scheme will require some modifications to the numerical procedures. To demonstrate the numerical procedures, a rectangular plate with a centrally located circular cutout is considered. The geometry and dimensions of the plate are shown in Fig.1. In this case, the total kinetic and potential energies can be obtained by subtracting the energies to the hole from the total energies for the rectangular plate.

$$T_{total} = \frac{1}{2}\dot{Q}^T M_p \dot{Q} - \frac{1}{2}\dot{Q}_c^T M_c \dot{Q}_c$$
$$U_{total} = \frac{1}{2}Q^T K_p Q - \frac{1}{2}Q_c^T K_c Q_c \tag{28}$$

Note that the boundary condition around the circular hole can be satisfied exactly, while the boundary condition along the rectangular outer edges of the plate must be handled with some numerical procedure. By using the coordinate transformation technique and geometrical relation between the Cartesian and polar coordinates, the displacement matching condition should be satisfied. Hence, the following condition should be satisfied inside the circular hole domain [30].

$$W_c(r,\theta) = W(\xi,\eta), \qquad \sum_{j=1}^{m_c} \varphi_{cj}(r,\theta) Q_{cj}(t) = \sum_{l=1}^{m} \varphi_l(r,\theta) Q_l(t) = \sum_{l=1}^{m} \phi_l(\xi)\psi_l(\eta) Q_l(t) \tag{29}$$

In this section , with the use of a weak solution and also with the use of orthogonality properties of trigonometric functions $\varphi_{ci}(r,\theta)$ and multiplying these functions in Eqs (29) and integration of these equations in the intervals of $0 < \theta < 2\pi$, the equations will be obtained in the form of polynomial functions based one finite degrees of R .

$$\sum_{j=1}^{m_c} \int_0^{2\pi}\int_0^R \varphi_{ci}(r,\theta)\varphi_{cj}(r,\theta) Q_{cj}(t)\, r\, dr\, d\theta = \sum_{l=1}^{m} \int_0^{2\pi}\int_0^R \varphi_{ci}(r,\theta)\phi_l(\xi)\psi_l(\eta) Q_l(t)\, r\, dr\, d\theta \tag{30}$$

Using the orthogonal property of $\varphi_{ci}(r,\theta)$, Eq (30) can be rewritten as:

$$Q_{ci}(t) = \sum_{l=1}^{m} \int_0^{2\pi}\int_0^R \varphi_{ci}(r,\theta)\phi_l(\xi)\psi_l(\eta) Q_l(t)\, r\, dr\, d\theta = \sum_{l=1}^{m} (F_c)_{il} Q_l(t) \tag{31}$$

Eq (31) can be expressed in matrix form:

$$Q_c = F_c Q \tag{32}$$

where F_c is a $m_c \times m$ transformation matrix [30]. By using the coordinate transformation technique and geometrical relation between the Cartesian and polar coordinates, the non-dimensionalized relationship can be written as:

$$\xi = \frac{l_1}{a} + \frac{r\cos(\theta)}{a}, \quad \eta = \frac{l_2}{b} + \frac{r\sin(\theta)}{b}. \tag{33}$$

with substituting Eq (32) into Eq (29) results in Eq (34).

$$T_{total} = \frac{1}{2}\dot{Q}^T M_p \dot{Q} - \frac{1}{2}\dot{Q}^T F_c^T M_c F_c \dot{Q} = \frac{1}{2}\dot{Q}^T M_{pc} \dot{Q}$$
$$U_{total} = \frac{1}{2}Q^T K_p Q - \frac{1}{2}Q^T F_c^T K_c F_c Q = \frac{1}{2}Q^T K_{pc} Q \tag{34}$$

By using the Eqs (7) and (18) and simplifying the Eq (34) can be written as follow:

$$M_{cp} = M_p - F_c^T M_c F_c = \rho hab\, \bar{M}_{cp} \tag{35a}$$

$$K_{cp} = K_p - F_c^T K_c F_c = \frac{Db}{a^3} \bar{K}_{cp} \tag{35b}$$

where

$$\bar{M}_{pc} = \bar{M}_p - (\pi\alpha\beta^2) F_c^T M_c F_c \tag{36a}$$

$$\bar{K}_{pc} = \bar{K}_p - \frac{\pi\alpha}{\beta^2} F_c^T \bar{K}_c F_c, \qquad \beta = \frac{R}{a} \tag{36b}$$

β which is the aspect ratio given as R/a .The eigenvalue equation for rectangular plate with circular hole can be derived as:

$$\left[\bar{K}_{cp} - \lambda^2 \bar{M}_{cp}\right]\{A\} = 0 \tag{37}$$

And the frequency parameter for rectangular plate with circular hole, λ is obtained by solving the generalized eigenvalue problem defined by Eq (20)

$$\det\left(\left[\bar{K}_{cp} - \lambda^2 \bar{M}_{cp}\right]\right) = 0 \tag{38}$$

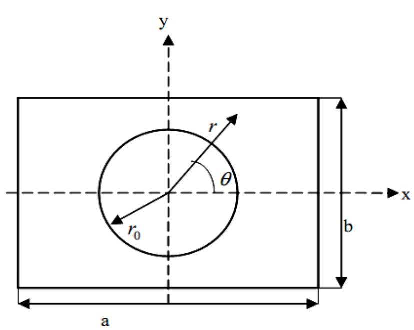

Fig. 1
Rectangular plate with circular central hole.

2.4 *Applying the Rayleigh-Ritz approach for rectangular plate having a circular hole with point support*

In this section, the transverse vibrations of rectangular plate with point supported have been studied and the natural frequencies are obtained by the classical Rayleigh-Ritz method. Strain energy of the supporting springs given by

$$U_{ps} = \frac{1}{2}\iint_A \sum_{s=1}^{N} k_s \delta(x - x_s)\delta(y - y_s) W^2(x_s, y_s, t) \tag{39}$$

where k_s the stiffness of the *sth* is spring and $W(x, y)$ is the transverse displacement. The kinetic energy can be expressed as:

$$T_{ps} = \frac{\omega^2}{2}\iint_A \sum_{s=1}^{N} m_s \delta(x - x_s)\delta(y - y_s) W^2(x_s, y_s, t) \tag{40}$$

By using the Eq (5) for the displacement of $W(x, y)$, the dynamic stiffness matrix of the plate will be derived as [28, 29]:

$$\Lambda_{ijmn} = \sum_{s=1}^{N} \varphi_i(x_s)\psi_j(x_s)\varphi_m(x_s)\psi_n(x_s) \tag{41}$$

where Λ_{ijmn} is the product of the basis functions and is evaluated where the springs and masses are located. In which case the stiffness matrix of the plate $K_{ps} = \Lambda_r \Lambda_r^T$. Where r is the rank of the support stiffness matrix. In the case of the plate with a elastic point supports, r=1, and

$$\Lambda = \left[\varphi_1(\xi_{s1})\psi_1(\eta_{s1}) \quad \varphi_1(\xi_{s1})\psi_2(\eta_{s1}) \qquad \varphi_m(\xi_{s1})\psi_{n-1}(\eta_{s1}) \quad \varphi_m(\xi_{s1})\psi_n(\eta_{s1})\right]^T \tag{42}$$

To determine the strain energy U_{PCS} and kinetic energy T_{PCS} of rectangular plate having a circular central hole with an elastic point supports, using the Eqs (39), (40) and Eq (34) results in Eq (43)

$$\begin{aligned} T_{PCS} &= \frac{1}{2}\dot{Q}^T M_p \dot{Q} + \frac{1}{2}\dot{Q}^T M_s \dot{Q} - \frac{1}{2}\dot{Q}^T F_c^T M_c F_c \dot{Q} = \frac{1}{2}\dot{Q}^T \bar{M}_{pcs}\dot{Q}, \\ U_{PCS} &= \frac{1}{2}Q^T K_p Q + \frac{1}{2}Q^T K_s Q - \frac{1}{2}Q^T F_c^T K_c F_c Q = \frac{1}{2}Q^T \bar{K}_{pcs} Q, \end{aligned} \tag{43}$$

where

$$\bar{K}_{pcs} = (\bar{K}_p + \sum_{s=1}^{N} \frac{k_s a^2 \alpha}{D}\Lambda) - \frac{\pi\alpha}{\beta^2} F_c^T \bar{K}_c F_c, \tag{44a}$$

$$\bar{M}_{pcs} = (\bar{M}_p + \frac{m_s}{\rho habM}\Lambda) - (\pi\alpha\beta^2) F_c^T M_c F_c. \tag{44b}$$

Above m_s is the *sth* discrete mass. The eigenvalue equation for rectangular plate having a circular hole with several point supports can be derived as:

$$\left[\bar{K}_{pcs} - \lambda^2 \bar{M}_{pcs}\right]\{A\} = 0, \tag{45}$$

And the frequency parameter for rectangular plate with circular hole, λ is obtained by solving the generalized eigenvalue problem defined by Eq (46)

$$\det\left(\left[\bar{K}_{pcs} - \lambda^2 \bar{M}_{pcs}\right]\right) = 0. \tag{46}$$

3 NUMERICAL RESULTS

In this section, numerical results are presented for the derived approximate closed-form results and compared to results generated using the previous and finite element method (FEM) for the elastically point supported plate.

Table 1
Fundamental natural frequency λ for simply supported square plate with a central hole

$\frac{2r_0}{a}$	ν								
	0.2			0.3			0.4		
	Present	Ref[16]	FEM	Present	Ref[16]	FEM	Present	Ref[16]	FEM
0.1	19.7409	19.61	19.81	19.7396	19.53	19.72	19.4521	19.3	19.49
0.15	19.7329	19.56	19.75	19.5795	19.39	19.58	19.2923	19.12	19.31
0.2	19.7358	19.6	19.79	19.4758	19.29	19.48	19.1883	18.91	19.09
0.25	19.7541	19.72	19.91	19.4796	19.32	19.51	19.1921	18.8	18.98
0.3	20.1234	20.01	20.21	19.7543	19.51	19.71	19.4668	18.89	19.07

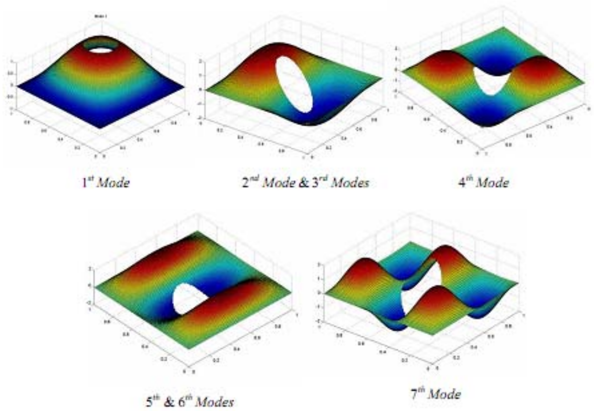

Fig. 2
Modes of vibration for simply supported plate with a circular central hole.

As it has been presented in Table 1, with increasing the radius of the hole, the frequency values are first decreased and then increases, this gained a special importance in optimizing the hole radius in analysis of these types of problems. In addition, the obtained results indicate that with increasing the value of Poisson's ratio, the frequency values would decrease. In Fig. 2, the first five modes of vibration for simply supported square plate with a central hole have been shown. In Table 2. also the frequency parameter of λ has been shown based on the different radius of the circular hole and different values of ν for a clamped square plate with a central hole and the results are very near to the results of reference [16], which indicate the accuracy of the suggested method. In this section also with increasing the hole radius, the values of frequencies are first decreased and then increased. Only with this difference that when we are using clamped square plate, the values of frequency parameter shows bigger values comparing to the case of simply supported plate.

Table 2
Fundamental natural frequency λ for clamped square plate with a central hole

$\frac{2r_0}{a}$	ν								
	0.2			0.3			0.4		
	Present	Ref[16]	FEM	Present	Ref[16]	FEM	Present	Ref[16]	FEM
0.1	35.9988	35.98	35.64	35.5256	35.83	35.46	35.2619	35.56	35.26
0.15	36.1068	36.28	35.89	35.6336	35.92	35.59	35.3699	35.49	35.16
0.2	36.6508	36.98	36.62	36.1776	36.41	36.12	35.9139	35.87	35.48
0.25	37.9834	36.83	37.73	37.5102	37.58	37.11	37.2465	36.82	36.38
0.3	38.9968	38.64	39.02	38.5236	39.39	38.95	38.2599	38.63	38.21

Table 3
Fundamental natural frequency λ of a simply supported square plate with a simply supported central hole.

$\frac{2r_0}{a}$	ν								
	0.2			0.3			0.4		
	Present	Ref[28]	FEM	Present	Ref[28]	FEM	Present	Ref[28]	FEM
0.1	52.62	52.65	53.14	50.64	50.75	51.14	48.79	48.9	49.28
0.2	58.95	59.02	59.25	56.73	56.78	57.02	54.67	54.71	54.89
0.3	70.08	70.11	70.64	67.44	67.53	67.98	64.98	65.07	65.52

In Fig.3, the first five modes of vibration for clamped square plate with a central hole have been shown.

In Table 3. also the frequency parameter of λ has been shown base on the different radius of the circular hole and different values of ν for a simply supported square plate with a simply supported central hole. In this section also with increasing the hole radius, the values of frequencies are increased.

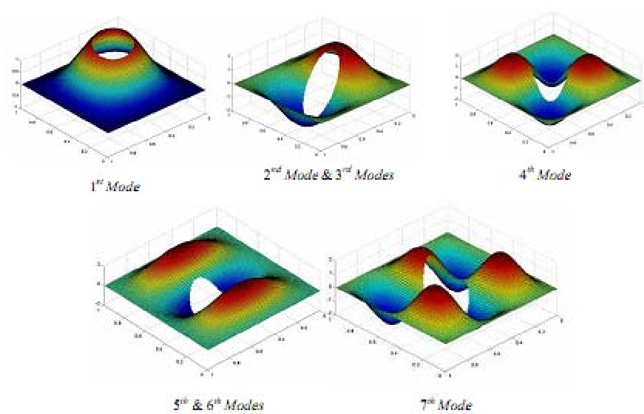

Fig. 3
Modes of vibration for clamped square plate with a circular central hole.

In Table 4. also the frequency parameter of λ has been shown based on the different radius of the circular hole and different values of ν for a simply supported square plate with a clamped central hole. In this section also with increasing the hole radius, the values of frequencies are increased.

Table 4
Fundamental natural frequency λ of a simply supported square plate with a clamped central hole.

$\frac{2r_0}{a}$	ν								
	0.2			0.3			0.4		
	Present	Ref[16]	FEM	Present	Ref[16]	FEM	Present	Ref[16]	FEM
0.1	62.68	62.79	63.31	60.32	60.42	60.92	58.12	58.22	58.70
0.2	75.92	76.14	76.75	73.12	73.27	73.85	70.46	70.60	71.16
0.3	93.06	93.16	93.99	89.55	89.64	90.44	86.29	86.38	87.15

Table 5
Fundamental natural frequency λ of a clamped rectangular plate with a circular central hole.

$\frac{a}{b}$	$2r_0/a$								
	0.2			0.3			0.4		
	present	Ref[16]	FEM	present	Ref[16]	FEM	present	Ref[16]	FEM
1	35.76	37.83	36.12	38.57	40.63	38.95	45.72	46.35	46.42
1.5	66.1	65.83	65.92	73.42	74.38	74.69	91.88	91.92	91.06
2	110.05	109.79	111.02	127.73	129.46	129.42	160.38	162.97	161.25

Table 6
Fundamental natural frequency λ of a simply supported rectangular plate with a circular central hole.

$\frac{a}{b}$	$2r_0/a$								
	0.2			0.3			0.4		
	present	Ref[16]	FEM	present	Ref[16]	FEM	present	Ref[16]	FEM
1	19.33	20.24	19.48	19.62	20.95	19.71	20.72	22.15	21.5
1.5	33.23	33.17	33.19	34.6	34.77	34.83	37.64	37.63	37.56
2	51.29	51.12	51.16	53.43	53.67	53.71	57.24	57.98	58.13

In Table 5. also the frequency parameter of λ has been shown based on the different radius of the circular hole and different values of a/b for a simply supported rectangular plate with a circular central hole. In this section also with increasing the hole radius and length of plate, the values of frequencies are increased.

In Table 6. also the frequency parameter of λ has been shown based on the different radius of the circular hole and different values of a/b for a simply supported rectangular plate with a circular central hole. In this section also with increasing the hole radius and length of plate, the values of frequencies are increased. In Table 7. the three first fundamental natural frequencies λ for rectangular plate with a corner point support Fig. 4 have been shown and depict a comparison of results between frequency coefficients available in reference [32, 33].

Table 7
Fundamental natural frequency λ for SFSF and CFCF rectangular plate with a corner point support.

n	Boundary condition for rectangular plate					
	SFSF			CFCF		
	present	Ref[32]	Ref[33]	present	Ref[32]	Ref[33]
1	9.745	9.61	9.7	23.88	15.17	23.93
2	17.31	17.32	16.81	23.62	15.38	23.29
3	30.56	30.61	30.44	39.36	39.4	39.29

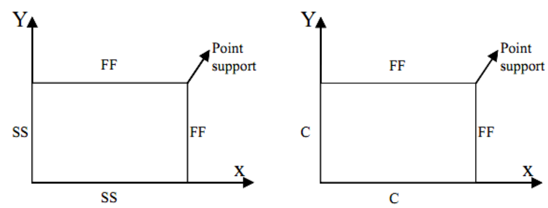

Fig. 4
Plate considered in the present study.

Table 8
Frequency parameters λ for a fully free square plate with a four point supports on the diagonals and $\beta = 0.2$

n	Plate without hole			Plate with central hole	
	present	Ref[27]	FEM[34]	present	FEM
1	13.5204	13.4682	13.4722	14.4838	14.3991
2	19.6103	19.5961	19.5963	20.7877	20.6544
3	26.9512	26.5941	26.5943	29.8700	29.4622

The final analysis model is a fully free square plate without any restraint on the boundary edges, as shown in Fig. 5 along with the new coordinate system. Four identical elastic supports, located symmetrically along the plate diagonals, are utilized to increase the fundamental natural frequency [27]. Table 8. lists the three first natural frequency parameter for free square plate with circular central hole.

The final results of the optimal solutions of the supports are given in Table 8 along with the result estimated by FEM [34,27]. With the respective optimal support solution, the fundamental natural frequency becomes a doubly repeated frequency for the desired frequency parameter 13.4682, and a triply repeated frequency for the desired frequency parameter 19.5961. In Fig. 6, some modes of vibration for A uniform square plate of fully free edges is supported by four elastic point supports on the diagonals (see Fig. 5) with a central hole have been shown.

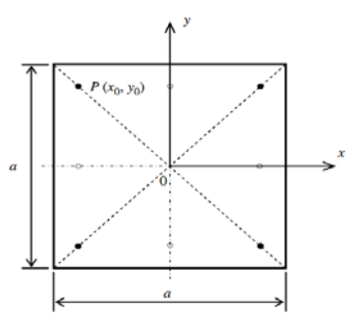

Fig. 5
A uniform square plate of fully free edges is supported by four elastic point supports on the diagonals (full points) or on the axes (hollow points).

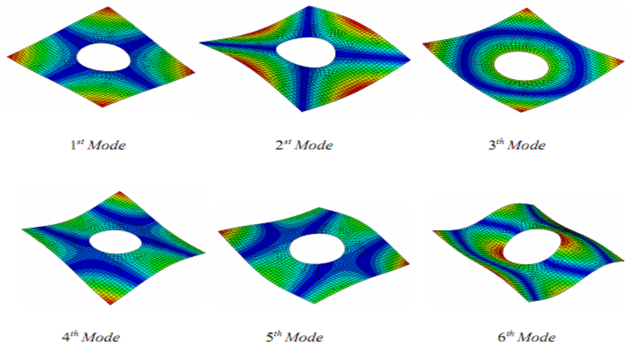

Fig. 6
Modes of vibration for a uniform square plate of fully free edges is supported by four elastic point supports on the diagonals with a central hole.

4 CONCLUSIONS

In this paper, the free vibration of rectangular plates with circular central hole for various boundary conditions was analyzed and natural frequencies were derived and compared with the reported results of other researchers. To solve the problem, it is necessary both Cartesian and polar coordinate system be used. For the validation, using the finite element method and modes of vibration for clamped and simply supported square plate with a central hole has been obtained. Comparison of the results obtained from the method used in this article, shows that the results are sufficiently accurate. Also to investigate the problem, long term and complex relationships, are not used and the problem is simply desired convergence is reached. In this study, the effects of increased the diameter of the hole on the natural frequencies were investigated and the optimum radius of the circular hole for different boundary conditions are obtained. The optimum value of the radius hole for simply supported square plate at $r_0 = 0.1a$ and in this case will have the least frequency, also the minimum value of the frequency for clamped square plate at $r_0 = 0.075a$. On the other hand, in this paper the free vibration of rectangular plate with circular central hole for point supported in different boundary condition was analyzed and natural frequencies were obtained and compared with the reported result by finite element method.

ACKNOWLEDGEMENTS

I would like to express my very great appreciation to Kashan University Faculty for his valuable and constructive suggestions during the planning and development of this research work. His willingness to give his time so generously has been very much appreciated. I would also like to thank the staff of the following organizations for enabling me to visit their offices to observe their daily operations: SUNIR Company

REFERENCES

[1] Monahan L.J., Nemergut P.J., Maddux G.E., 1970, Natural frequencies and mode shapes of plates with interior cut-outs, *The Shock and Vibration Bulletin* **41**:37-49.

[2] Paramasivam P., 1973, Free vibration of square plates with square opening, *Journal of Sound and Vibration* 30:173-178.

[3] Aksu G., Ali R., 1976, Determination of dynamic characteristics of rectangular plates with cut-outs using a finite difference formulation, *Journal of Sound and Vibration* **44**:147-158.

[4] Rajamani A., Prabhakaran R., 1977, Dynamic response of composite plates with cut-outs, *Journal of Sound and Vibration* **54**:549-564.

[5] Rajamani A., Prabhakaran R., 1977, Dynamic response of composite plates with cut-outs, *Journal of Sound and Vibration* **54**:565-576.

[6] Ali R., Atwal S.J., 1980, Prediction of natural frequencies of vibration of rectangular plates with rectangular cutouts, *Computers and Structures* **12**(9):819-823.

[7] Lam K.Y., Hung K.C, Chow S.T., 1989, Vibration analysis of plates with cut-outs by the modified rayleigh-ritz method, *Applied Acoustics* **28**: 49-60.

[8] Lam K.Y., Hung K.C., 1990, Vibration study on plates with stiffened openings using orthogonal polynomials and partitioning method, *Computers and Structures* **37**:295-301.

[9] Laura P.A., Romanelli E., Rossi R.E., 1997, Transverse vibrations of simply-supported rectangular plates with rectangular cutouts, *Journal of Sound and Vibration* **202**(2):275-283.

[10] Sakiyama T., Huang M., Matsuda H., Morita C., 2003, Free vibration of orthotropic square plates with a square hole, *Journal of Sound and Vibration* **259**(1):63-80.

[11] Joga-Rao C.V., Pickett G., 1961, Vibrations of plates of irregular shapes and plates with holes, *Journal of the Aeronautical Society of India* **13**(3):83-88.

[12] Kumai T., 1952, The flexural vibrations of a square plate with a central circular hole, *Proceedings of 2nd Japan National Congress on Applied Mechanics* 339-342.

[13] Hegarty R.F., Ariman T., 1975, Elasto-dynamic analysis of rectangular plates with circular holes, *International Journal of Solids and Structures* **11**:895-906.

[14] Eastep F.E., Hemmig F.G., 1978, Estimation of fundamental frequency of non-circular plates with free,circular cutouts, *Journal of Sound and Vibration* **56**(2):155-165.

[15] Nagaya K., 1952, Transverse vibration of a plate having an eccentric inner boundary, *Journal of Applied Mechanics* **18** (3):1031-1036.

[16] Nagaya K., 1980, Transverse vibration of a rectangular plate with an eccentric circular inner boundary, *International Journal of Solids and Structures* **16**:1007-1016.

[17] Lee H.S., Kim K.C., 1984, Transverse vibration of rectangular plates having an inner cutout in water, *Journal of the Society of Naval Architects of Korea* **21**(1):21-34.

[18] Kim K.C., Han S.Y., Jung J.H., 1987, Transverse vibration of stiffened rectangular plates having an inner cutout. *Journal of the Society of Naval Architects of Korea* **24**(3):35-42.

[19] Avalos D.R., Laura P.A., , Transverse vibrations of simply supported rectangular plates with two rectangular cutouts, *Journal of Sound and Vibration* **267**:967-977.

[20] Lee H.S., Kim K.C., 1984, Transverse vibration of rectangular plates having an inner cutout in water, *Journal of the Society of Naval Architects of Korea* **21**(1):21-34.

[21] Khurasia H.B., Rawtani S., 1978, Vibration analysis of circular plates with eccentric hole, *Journal of Applied Mechanics* **45**(1):215-217.

[22] Lin W.H., 1982, Free transverse vibrations of uniform circular plates and membranes with eccentric holes, *Journal of Sound and Vibration* **81**(3): 425-433.

[23] Laura P.A., Masia U., Avalos D.R.,2006, Small amplitude, transverse vibrations of circular plates elastically restrained against rotation with an eccentric circular perforation with a free edge, *Journal of Sound and Vibration* **292**:1004-1010.

[24] Cheng L., Li Y.Y., Yam L.H., 2003, Vibration analysis of annular-like plates, *Journal of Sound and* Vibration **262**: 1153-1170.

[25] Lee W.M., Chen J.T, Lee Y.T.,2007, Free vibration analysis of circular plates with multiple circular holes using indirect BIEMs, *Journal of Sound and Vibration* **304**:811-830.

[26] Zhong H., Yu T., 2007, Flexural vibration analysis of an eccentric annular mindlin plate, *Archive of Applied Mechanics* **77**:185-195.

[27] Wang D., Yang Z.C., Yu Z.G.,2010, Minimum stiffness location of point support for control of fundamental natural frequency of rectangular plate by Rayleigh–Ritz method, *Journal of Sound and Vibration* **329**:2792-2808.

[28] Joseph Watkins R., Barton Jr O., 2010, Characterizing the vibration of an elastically point supported rectangular plate using eigensensitivity analysis, *Thin-Walled Structures* **48**:327-333.

[29] Dozio L., 2011, On the use of the trigonometric ritz method for general vibration analysis of rectangular kirchhoff plates, *Thin-Walled Structures* **49**:129-144.

[30] Kwak M.K., Han S.,2007, Free vibration analysis of rectangular plate with a hole by means of independent coordinate coupling method, *Journal of Sound and Vibration* **306**:12-30.

[31] Saeedi K., Leo A., 2012, Vibration of circular plate with multiple eccentric circular perforations by the Rayleigh-Ritz method, *Journal of Mechanical Science and Technology* **26** (5):1439-1448.

[32] Fan S.C., Cheung Y.K., 1984, Flexural free vibrations of rectangular plates with complex support conditions, *Journal of Sound and Vibration* **93**:81-94.

[33] Utjes J.C., Laura P.A., 1984, Vibrations of thin elastic plates with point supports: a comparative study, *Second National Meeting of Users of the Method of Finite Elements.*

[34] Wang D., Jiang J.S., Zhang W.H., 2004, Optimization of support positions to maximize the fundamental frequency of structures, *International Journal for Numerical Methods in Engineering* **61**:1584-1602.