

A New Approach to the Study of Transverse Vibrations of a Rectangular Plate Having a Circular Central Hole

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ABSTRACT

In this study, the analysis of transverse vibrations of rectangular plate with circular central hole with different boundary conditions is studied and the natural frequencies and natural modes of a rectangular plate with circular hole have been obtained. To solve the problem, it is necessary to use both Cartesian and polar coordinate system. The complexity of the method is to apply an appropriate model, which can solve the problem of transverse vibrations of a plate. So, it has been tried that the functions of the deflection of plate, in the form of polynomial functions proportionate with finite degrees, to be replaced by Bessel function, which is used in the analysis of the vibrations of a circular plate. Then with the help of a semi-analytical method and orthogonality properties of the eliminated position angle, without any need to analyze so many points on the edges of the rectangular plate, we can prevent the coefficients matrix from becoming so much large as well as the equations from becoming complicated. The above mentioned functions will lead to reducing the calculation time and simplifying the equations as well as speeding up the convergence.

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1 INTRODUCTION

RECTANGULAR plate with a rectangular or a circular hole has been widely used as a substructure for ship, airplane, and plant. Uniform circular, annular and rectangular plates have been also widely used as structural components for various industrial applications and their dynamic behaviors can be described by exact solutions. However, the vibration characteristics of a rectangular plate with an eccentric circular hole cannot be analyzed easily [1-5]. The vibration characteristics of a rectangular plate with a hole can be solved by either the Rayleigh-Ritz method or the finite element method. However, it cannot be easily applied to the case of a rectangular plate with a circular hole since the admissible functions for the rectangular hole domain do not permit closed-form integrals. Many studies have been done on the subject, some of which are mentioned in this section. Monahan et al. [1] applied the finite element method to a clamped rectangular plate with a rectangular hole and verified the numerical results by experiments. Paramasivam [2] used the finite difference method for a simply-supported and clamped rectangular plate with a rectangular hole. There are many research works concerning plate with a single hole but a few works on plate with multiple holes. Aksu and Ali [3] also used the finite difference method to analyze a rectangular plate with more than two holes. Rajamani and Prabhakaran [4] assumed that the effect of a hole is equivalent to an externally applied loading and carried out a numerical analysis based on this assumption for a composite plate. Rajamani and Prabhakaran [5] investigated the effect of a hole on the natural vibration

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characteristics of isotropic and orthotropic plates with simply-supported and clamped boundary conditions. Ali and Atwal [6] applied the Rayleigh-Ritz method to a simply-supported rectangular plate with a rectangular hole, using the static deflection curves for a uniform loading as admissible functions. Lam et al. [7] divided the rectangular plate with a hole into several sub areas and applied the modified Rayleigh-Ritz method. Lam and Hung [8] applied the same method to a stiffened plate. Laura et al. [9] calculated the natural vibration characteristics of a simply-supported rectangular plate with a rectangular hole by the classical Rayleigh-Ritz method. Sakiyama et al. [10] analyzed the natural vibration characteristics of an orthotropic plate with a square hole by means of the Green function assuming the hole as an extremely thin plate.

The vibration analysis of a rectangular plate with a circular hole does not lend an easy approach since the geometry of the hole is not the same as the geometry of the rectangular Plate. Takahashi used the classical Rayleigh-Ritz method after deriving the total energy by subtracting the energy of the hole from the energy of the whole plate. He employed the eigenfunctions of a uniform beam as admissible functions. Joga-Rao and Pickett [11] proposed the use of algebraic polynomial functions and biharmonic singular functions. Kumai [12] Hegarty [13], Eastep and Hemmig [14], and Nagaya [15-16] used the point-matching method for the analysis of a rectangular plate with a circular hole. The point-matching method employed the polar coordinate system based on the circular hole and the boundary conditions were satisfied along the points located on the sides of the rectangular plate. Lee and Kim [17] carried out vibration experiments on the rectangular plates with a hole in air and water. Kim et al. [18] performed the theoretical analysis on a stiffened rectangular plate with a hole. Avalos and Laura [19] calculated the natural frequency of a simply-supported rectangular plate with two rectangular holes using the Classical Rayleigh-Ritz method. Lee et al. [20] analyzed a square plate with two collinear circular holes using the classical Rayleigh-Ritz method.

A circular plate with an eccentric circular hole has been treated by various methods. Khurasia and Rawtani [21] studied the effect of the eccentricity of the hole on the vibration characteristics of the circular plate by using the triangular finite element method. Lin [22] used an analytical method based on the transformation of Bessel Functions to calculate the free transverse vibrations of uniform circular plates and membranes with eccentric holes. Laura et al. [23] applied the Rayleigh-Ritz method to circular plates restrained against rotation with an eccentric circular perforation with a free edge. Cheng et al. [24] used the finite element analysis code, Nastran, to analyze the effects of the hole eccentricity, hole size and boundary condition on the vibration modes of annular-like plates. Lee et al. [25] used an indirect formulation in conjunction with degenerate kernels and Fourier series to solve the natural frequencies and modes of circular plates with multiple circular holes and verified the finite element solution by using ABAQUS. Zhong and Yu. [26] formulated a weak-form quadrature element method to study the flexural vibrations of an eccentric annular Mindlin plate. As it was mentioned earlier, in most of the researches done in this field, Rayleigh-Ritz method and numerical methods have been used and with the help of reducing the hole energy comparing to the energy of the whole rectangular plate, the problem has been analyzed. Also for studying the issues considering the position conditions and angle, the quantity of so many points in the edges of the rectangular plate have been used.

In this study, the analysis of transverse vibrations of rectangular plate with circular central hole with different boundary conditions is studied and with the help of a semi-analytical method, the natural frequencies and natural modes of a rectangular plate with circular hole have been obtained. In this method, a simple polynomial functions, the desired frequency range, which can replace the Bessel functions will be used, and convergence the problem will be obtained easily.

2 FORMULATION OF THE PROBLEM

According to the classical thin plate theory, the free harmonic vibration of a thin plate with a constant thickness h is governed by the differential equation

$$\nabla^4 \bar{W} + \frac{\rho h}{D} \frac{\partial^2 \bar{W}}{\partial t^2} = p(x, y, z, t) \quad (1)$$

where both p and \bar{W} are functions of time, as well as space, ρ is the mass density of the material, and D is the plate flexural rigidity defined as:

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad (2)$$

Here, E is Young's modulus and ν is Poisson's ratio. For forced vibration $p(x, y, z, t)$ causes the dynamic response, and can vary from a harmonic oscillation to an intense one time impact. Thus for study the natural vibrations $p(x, y, z, t)$ is set equal to zero, and the governing equation becomes the following homogeneous equation:

$$\nabla^4 \bar{W} + \frac{\rho h}{D} \frac{\partial^2 \bar{W}}{\partial t^2} = 0. \quad (3)$$

Let us consider a rectangular plate of uniform thickness containing a central circular hole with side lengths a in the X direction and b in the Y direction as shown in Fig. 1.

If one assumes a solution of Eq. (3) in the form

$$\bar{W} = W \cos \omega t \quad (4)$$

where ω is the natural frequency and W is the deflection amplitude functions. Then Eq. (3) becomes:

$$D \nabla^4 W = \omega^2 \rho h W. \quad (5)$$

The n th vibration mode of Eq. (5) in polar coordinates is [27]

$$W = \sum_{n=0}^{\infty} \{A_n J_n(\lambda_n R) + B_n Y_n(\lambda_n R) + C_n I_n(\lambda_n R) + D_n K_n(\lambda_n R)\} \cos n\theta \quad (6)$$

$$\sum_{n=1}^{\infty} \{A_n^* J_n(\lambda_n R) + B_n^* Y_n(\lambda_n R) + C_n^* I_n(\lambda_n R) + D_n^* K_n(\lambda_n R)\} \sin n\theta.$$

where $R = r/l$ a non-dimensionalized is coordinated and l has the dimensions of length. Also in Eq. (6)

$$\lambda = \omega a^2 \sqrt{\frac{\rho h}{D}} \quad (7)$$

In Eq. (6) A_n, \dots, D_n and A_n^*, \dots, D_n^* are constants to be determined, J_n and I_n are the Bessel function and the modified Bessel function of the first kind, Y_n and K_n are Bessel function and the modified Bessel function of the second kind of order n , respectively. The boundary conditions at $X = \pm l_1$ are: (see Fig.1)

$$W = 0, \quad \frac{\partial W}{\partial X} = 0. \quad (8a)$$

and at $Y = \pm l_2$ are:

$$W = 0, \quad \frac{\partial W}{\partial Y} = 0. \quad (9a)$$

where $X = \frac{x}{l}, Y = \frac{y}{l}$. Also, $l_1 = \frac{a}{2l}, l_2 = \frac{b}{2l}$. Boundary condition (8a) and (9a) are to be used for a plate with clamped edges. If the plate is simply supported, the boundary conditions at $X = \pm l_1$ are shown as:

$$W = 0, \quad \frac{\partial^2 W}{\partial Y^2} = 0. \tag{8b}$$

and at $Y = \pm l_2$ are:

$$W = 0, \quad \frac{\partial^2 W}{\partial X^2} = 0 \tag{9b}$$

and for free edges at $X = \pm l_1$:

$$\frac{\partial^2 W}{\partial X^2} + \nu \frac{\partial^2 W}{\partial Y^2} = 0, \quad \frac{\partial^3 W}{\partial X^3} + (2-\nu) \frac{\partial^3 W}{\partial X \partial Y^2} = 0. \tag{8c}$$

and at $Y = \pm l_2$:

$$\frac{\partial^2 W}{\partial Y^2} + \nu \frac{\partial^2 W}{\partial X^2} = 0, \quad \frac{\partial^3 W}{\partial Y^3} + (2-\nu) \frac{\partial^3 W}{\partial Y \partial X^2} = 0. \tag{9c}$$

Since the circular hole is to be free of all applied stress, the boundary condition to be satisfied along the edge of the hole at $R = R_0$ are:

$$M_R = 0, \quad Q_R - \frac{1}{R} \frac{\partial M_{R\theta}}{\partial \theta} = 0. \tag{10a}$$

and the other boundary conditions along the inner edge are

For the clamped edge:

$$at \quad R = R_0 : \quad W = \frac{\partial W}{\partial R} = 0 \tag{10b}$$

For the simply supported edge:

$$at \quad R = R_0 : \quad W = M_R = 0 \tag{10c}$$

where M_R is the bending moment normal to the hole, $M_{R\theta}$ is the twisting moment in the same plane, and the Q_R is the shear force acting at the edge of the hole. In terms of the deflection W , Eq. (10a) become:

$$R = R_0 : \quad \frac{\partial^2 W}{\partial R^2} + \frac{\nu}{R^2} \frac{\partial^2 W}{\partial \theta^2} + \frac{\nu}{R} \frac{\partial W}{\partial R} = 0, \quad R = R_0 : \quad \frac{\partial^3 W}{\partial R^3} + \frac{2-\nu}{R^2} \frac{\partial^3 W}{\partial R \partial \theta^2} - \frac{3-\nu}{R^3} \frac{\partial^2 W}{\partial R^2} + \frac{1}{R} \frac{\partial^2 W}{\partial R^2} - \frac{1}{R^2} \frac{\partial W}{\partial R} = 0. \tag{11}$$

The solutions (6), along with the boundary conditions (8) through (10), uniquely determine the stress field. Because of the rectangular boundaries, however, the boundary conditions cannot be satisfied identically on all the boundaries.

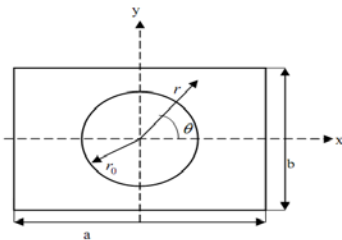


Fig. 1
Rectangular plate with circular central hole.

2.1 Analysis

In this section, the choice of coordinate axes at the center of a circular hole (see Fig.1) and using boundary condition on the circular plate and development condition by choosing different points (According the radius and angle positioning of points) on the edges of rectangular plate, the problem will be analyzed, and using the orthogonality properties of weak solution the relationship will be expanded and then using the boundary condition on the edges of rectangular plate, the natural frequencies are obtained.

Substituting the Eq. (6) into the boundary condition (11), the following equations are obtained

$$\alpha_1 A_n + \alpha_2 B_n + \alpha_3 C_n + \alpha_4 D_n = 0, \quad (12a)$$

$$\beta_1 A_n + \beta_2 B_n + \beta_3 C_n + \beta_4 D_n = 0, \quad (12b)$$

$$\alpha_1 A_n^* + \alpha_2 B_n^* + \alpha_3 C_n^* + \alpha_4 D_n^* = 0, \quad (12c)$$

$$\beta_1 A_n^* + \beta_2 B_n^* + \beta_3 C_n^* + \beta_4 D_n^* = 0, \quad (12d)$$

where

$$\alpha_1 = J_n'' + \frac{\nu}{R_0} J_n' - \frac{\nu n^2}{R_0^2} J_n, \quad \alpha_2 = Y_n'' + \frac{\nu}{R_0} Y_n' - \frac{\nu n^2}{R_0^2} Y_n, \quad \alpha_3 = I_n'' + \frac{\nu}{R_0} I_n' - \frac{\nu n^2}{R_0^2} I_n, \quad \alpha_4 = K_n'' + \frac{\nu}{R_0} K_n' - \frac{\nu n^2}{R_0^2} K_n. \quad (13)$$

and

$$\begin{aligned} \beta_1 &= J_n''' + \frac{1}{R_0} J_n'' - \frac{1}{R_0^2} (1+n^2(2-\nu)) J_n' + \frac{3-\nu}{R_0^3} n^2 J_n, \quad \beta_2 = Y_n''' + \frac{1}{R_0} Y_n'' - \frac{1}{R_0^2} (1+n^2(2-\nu)) Y_n' + \frac{3-\nu}{R_0^3} n^2 Y_n, \\ \beta_3 &= I_n''' + \frac{1}{R_0} I_n'' - \frac{1}{R_0^2} (1+n^2(2-\nu)) I_n' + \frac{3-\nu}{R_0^3} n^2 I_n, \quad \beta_4 = K_n''' + \frac{1}{R_0} K_n'' - \frac{1}{R_0^2} (1+n^2(2-\nu)) K_n' + \frac{3-\nu}{R_0^3} n^2 K_n. \end{aligned} \quad (14)$$

As it has been shown in Eq. (6), deflection of the intended plate, can be expressed in terms of Bessel functions of the first and second kind. Due to the properties of the Bessel functions and regardless of terms with high degrees in Eq.(6) and also obtained frequencies with the use of Finite Element method, in this section it has been tried to acquire the natural frequencies and the mode shapes of the rectangular plate with a central hole, with the use of polynomial functions proportionate with finite degrees in the intended frequency limits instead of the mentioned Bessel functions. Bessel functions of the first kind, denoted as $J_n(R)$, are solutions of Bessel's differential equation that are finite at the origin ($R = 0$) for integer n , and diverge as R approaches zero for negative non-integer n . The solution type (e.g., integer or non-integer) and normalization of $J_n(R)$ are defined by its properties below. It is possible to define the function by its Taylor series expansion around $R = 0$

$$J_n(R) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+n+1)} \left(\frac{1}{2}\right)^{2m+n} = P_n(R) \quad (15)$$

where $\Gamma(Z)$ is the gamma function. The series expansion for $I_n(R)$ is thus similar to that for $J_n(R)$, but without the alternating $(-1)^m$ factor.

$$I_n(R) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m+n+1)} \left(\frac{1}{2}\right)^{2m+n} = S_n(R) \quad (16)$$

The series expansion for $Y_n(R)$ and $K_n(R)$ using a series expansion of Bessel functions $J_n(R)$ and $I_n(R)$ will be obtained easily. Here, $P_n(R), Q_n(R), S_n(R), T_n(R)$ are the polynomials with a limited degree, will be sought in the form of series expansions and in the desired frequency range, will be replaced the Bessel functions $J_n(R), Y_n(R), I_n(R), K_n(R)$, respectively. Other relationships, by substituting the polynomials functions and

simplifying the equations will be obtained. For example, if $n = 1$, the following proposed polynomials, can be replaced by the Bessel functions:

$$\begin{aligned}
 J_1 &= P_1(R) = 1 - \frac{1}{4}\lambda^2 R^2 + \frac{1}{64}\lambda^4 R^4 - \frac{1}{2304}\lambda^6 R^6, \\
 Y_1 &= Q_1(R) = \frac{2\log(\lambda R/2)}{\pi} + \frac{2\Gamma}{\pi} + \left(\frac{2-2\Gamma}{4\pi}\lambda^2 - \frac{\log(\lambda R/2)}{2\pi}\lambda^2 \right) R^2, \\
 I_1 &= S_1(R) = 1 + \frac{1}{4}\lambda^2 R^2 + \frac{1}{64}\lambda^4 R^4 + \frac{1}{2304}\lambda^6 R^6, \\
 K_1 &= T_1(R) = -\log(\lambda R/2) - \Gamma + \left(\frac{2-2\Gamma}{8}\lambda^2 - \frac{\log(\lambda R/2)}{4}\lambda^2 \right) R^2 + \left(\frac{3-2\Gamma}{128}\lambda^4 - \frac{\log(\lambda R/2)}{64}\lambda^4 \right) R^4,
 \end{aligned} \tag{17}$$

where Γ the size is the gamma function ($\Gamma = .57722$). Note that the boundary condition around the circular hole can be satisfied exactly, while the boundary condition along the rectangular outer edges of the plate must be handled with some numerical procedure. Eq. (12a) and (12b) can be solved for A_n and B_n in terms of C_n and D_n .

$$A_n = \gamma_1 C_n + \gamma_2 D_n, \quad B_n = \gamma_3 C_n + \gamma_4 D_n, \tag{18}$$

where

$$\gamma_1 = \frac{\alpha_2\beta_3 - \alpha_3\beta_2}{\alpha_1\beta_2 - \alpha_2\beta_1}, \quad \gamma_2 = \frac{\alpha_2\beta_4 - \alpha_4\beta_2}{\alpha_1\beta_2 - \alpha_2\beta_1}, \quad \gamma_3 = \frac{\alpha_3\beta_1 - \alpha_1\beta_3}{\alpha_1\beta_2 - \alpha_3\beta_1}, \quad \gamma_4 = \frac{\alpha_4\beta_1 - \alpha_1\beta_4}{\alpha_1\beta_2 - \alpha_3\beta_1}. \tag{19}$$

Eq. (12c) and (12d) show that the same relationship exists for A_n^*, B_n^*, C_n^* and D_n^*

$$A_n^* = \gamma_1 C_n^* + \gamma_2 D_n^*, \quad B_n^* = \gamma_3 C_n^* + \gamma_4 D_n^*. \tag{20}$$

Substituting the Eq.(6) into the boundary condition(8) and(9), and using of Eq.(18) and (20), the following equation will be obtained.

$$\sum_{n=0}^{\infty} \left\{ C_n [(S_n + \gamma_1 P_n + \gamma_3 Q_n) \cos n\theta] + D_n [(T_n + \gamma_2 P_n + \gamma_4 Q_n) \cos n\theta] \right. \\ \left. + C_n^* [(S_n + \gamma_1 P_n + \gamma_3 Q_n) \sin n\theta] + D_n^* [(T_n + \gamma_2 P_n + \gamma_4 Q_n) \sin n\theta] \right\} = 0. \tag{21}$$

Eq. (21), considering the expressed boundary conditions in Eqs. (8) and (9) is applicable at $X = \pm l_1, Y = \pm l_2$ for either simple supported or clamped edges.

By using the coordinate transformation technique and geometrical relation between the Cartesian and polar coordinates, and substituting the Eq. (6) into the boundary condition (8) and (9) and simplifying the relations, leads to achieved Eq. (22).

$$\sum_{n=0}^{\infty} \left\{ C_n \left[(S'_n + \gamma_1 P'_n + \gamma_3 Q'_n) \cos n\theta \cos \theta + (S_n + \gamma_1 P_n + \gamma_3 Q_n) \frac{n}{R} \sin n\theta \sin \theta \right] + \right. \\ \left. + D_n \left[(T'_n + \gamma_2 P'_n + \gamma_4 Q'_n) \cos n\theta \cos \theta + (T_n + \gamma_2 P_n + \gamma_4 Q_n) \frac{n}{R} \sin n\theta \sin \theta \right] \right. \\ \left. + C_n^* \left[(S'_n + \gamma_1 P'_n + \gamma_3 Q'_n) \sin n\theta \cos \theta - (S_n + \gamma_1 P_n + \gamma_3 Q_n) \frac{n}{R} \cos n\theta \sin \theta \right] \right. \\ \left. + D_n^* \left[(T'_n + \gamma_2 P'_n + \gamma_4 Q'_n) \sin n\theta \cos \theta - (T_n + \gamma_2 P_n + \gamma_4 Q_n) \frac{n}{R} \cos n\theta \sin \theta \right] \right\} = 0. \tag{22}$$

Eq. (22) is applied at $X = \pm l_1$ for clamped edges. For either simply supported and free edges at $X = \pm l_1$ Eq. (23) can be used as follow

$$\sum_{n=0}^{\infty} \{C_n [\Psi_1] + D_n [\Psi_2] + C_n^* [\Psi_3] + D_n^* [\Psi_4]\} = 0, \quad (23)$$

where

$$\begin{aligned} \Psi_1 = & (\cos^2 \theta + \nu \sin^2 \theta) \cos n\theta (S_n'' + \gamma_1 P_n'' + \gamma_3 Q_n'') + \frac{1}{R} (n(1-\nu) \sin 2\theta \sin n\theta) + \cos n\theta (\sin^2 \theta + \nu \cos^2 \theta) (S_n' + \gamma_1 P_n' + \gamma_3 Q_n') \\ & - \frac{n}{R^2} (n \cos n\theta (\sin^2 \theta + \nu \cos^2 \theta) + (1-\nu) \sin 2\theta \sin n\theta) (S_n + \gamma_1 P + \gamma_3 Q). \end{aligned} \quad (23a)$$

$$\begin{aligned} \Psi_2 = & \cos n\theta (\cos^2 \theta + \nu \sin^2 \theta) (T_n'' + \gamma_2 P_n'' + \gamma_4 Q_n'') + \frac{1}{R} (n(1-\nu) \sin 2\theta \sin n\theta) + \cos n\theta (\sin^2 \theta + \nu \cos^2 \theta) (T_n' + \gamma_2 P_n' + \gamma_4 Q_n') \\ & - \frac{n}{R^2} (n \cos n\theta (\sin^2 \theta + \nu \cos^2 \theta) + (1-\nu) \sin 2\theta \sin n\theta) (T_n + \gamma_2 P_n + \gamma_4 Q_n) \end{aligned} \quad (23b)$$

$$\begin{aligned} \Psi_3 = & (\cos^2 \theta + \nu \sin^2 \theta) \sin n\theta (S_n'' + \gamma_1 P_n'' + \gamma_3 Q_n'') + \frac{1}{R} (\sin n\theta (\sin^2 \theta + \nu \cos^2 \theta) - n(1-\nu) \sin 2\theta \cos n\theta) (S_n' + \gamma_1 P_n' + \gamma_3 Q_n') \\ & + \frac{n}{R^2} ((1-\nu) \sin 2\theta \cos n\theta - n \sin \theta (\sin^2 \theta + \nu \cos^2 \theta)) (S_n + \gamma_1 P + \gamma_3 Q) \end{aligned} \quad (23c)$$

$$\begin{aligned} \Psi_4 = & (\cos^2 \theta + \nu \sin^2 \theta) \sin n\theta (T_n'' + \gamma_2 P_n'' + \gamma_4 Q_n'') + \frac{1}{R} (\sin n\theta (\sin^2 \theta + \nu \cos^2 \theta) - n(1-\nu) \sin 2\theta \cos n\theta) (T_n' + \gamma_2 P_n' + \gamma_4 Q_n') \\ & + \frac{n}{R^2} ((1-\nu) \sin 2\theta \cos n\theta - n \sin \theta (\sin^2 \theta + \nu \cos^2 \theta)) (T_n + \gamma_2 P_n + \gamma_4 Q_n) \end{aligned} \quad (23d)$$

For clamped edges at $Y = \pm l_2$ Eq. (24) can be used as follows.

$$\sum_{n=0}^{\infty} \left\{ \begin{aligned} & C_n \left[(S_n' + \gamma_1 P_n' + \gamma_3 Q_n') \cos n\theta \sin \theta + \frac{n}{R} (S_n + \gamma_1 P_n + \gamma_3 Q_n) \sin n\theta \cos \theta \right] \\ & + D_n \left[(T_n' + \gamma_2 P_n' + \gamma_4 Q_n') \cos n\theta \sin \theta - \frac{n}{R} (T_n + \gamma_2 P_n + \gamma_4 Q_n) \sin n\theta \cos \theta \right] \\ & + C_n^* \left[(S_n' + \gamma_1 P_n' + \gamma_3 Q_n') \sin n\theta \sin \theta + \frac{n}{R} (S_n + \gamma_1 P_n + \gamma_3 Q_n) \cos n\theta \cos \theta \right] \\ & + D_n^* \left[(T_n' + \gamma_2 P_n' + \gamma_4 Q_n') \sin n\theta \sin \theta + \frac{n}{R} (T_n + \gamma_2 P_n + \gamma_4 Q_n) \cos n\theta \cos \theta \right] \end{aligned} \right\} = 0. \quad (24)$$

For either simply supported and free edges at $Y = \pm l_2$ Eq. (25) can be used as follow

$$\sum_{n=0}^{\infty} \{C_n [\Lambda_1] + D_n [\Lambda_2] + C_n^* [\Lambda_3] + D_n^* [\Lambda_4]\} = 0, \quad (25)$$

where

$$\begin{aligned} \Lambda_1 = & (\sin^2 \theta + \nu \cos^2 \theta) \cos n\theta (S_n'' + \gamma_1 P_n'' + \gamma_3 Q_n'') + \frac{1}{R} (n(\nu-1) \sin 2\theta \sin n\theta) + \cos n\theta (\cos^2 \theta + \nu \sin^2 \theta) (S_n' + \gamma_1 P_n' + \gamma_3 Q_n') \\ & - \frac{n}{R^2} (n \cos n\theta (\cos^2 \theta + \nu \sin^2 \theta) - (1-\nu) \sin 2\theta \sin n\theta) (S_n + \gamma_1 P + \gamma_3 Q). \end{aligned} \quad (25a)$$

$$\begin{aligned} \Lambda_2 = & \cos n\theta (\sin^2 \theta + \nu \cos^2 \theta) (T_n'' + \gamma_2 P_n'' + \gamma_4 Q_n'') + \frac{1}{R} (n(\nu-1) \sin 2\theta \sin n\theta) + \cos n\theta (\cos^2 \theta + \nu \sin^2 \theta) (T_n' + \gamma_2 P_n' + \gamma_4 Q_n') \\ & - \frac{n}{R^2} (n \cos n\theta (\cos^2 \theta + \nu \sin^2 \theta) - (1-\nu) \sin 2\theta \sin n\theta) (T_n + \gamma_2 P_n + \gamma_4 Q_n). \end{aligned} \quad (25b)$$

$$\Lambda_3 = \sin n\theta(\sin^2 \theta + \nu \cos^2 \theta)(S_n'' + \gamma_1 P_n'' + \gamma_3 Q_n'') + \frac{1}{R}(\sin n\theta(\cos^2 \theta + \nu \sin^2 \theta) + n(1-\nu) \sin 2\theta \cos n\theta)(S_n' + \gamma_1 P_n' + \gamma_3 Q_n') - \frac{n}{R^2}(n \sin n\theta(\cos^2 \theta + \nu \sin^2 \theta) + (1-\nu) \sin 2\theta \cos \theta \cos n\theta)(S_n + \gamma_1 P + \gamma_3 Q). \quad (25c)$$

$$\Lambda_4 = \sin n\theta(\sin^2 \theta + \nu \cos^2 \theta)(T_n'' + \gamma_2 P_n'' + \gamma_4 Q_n'') + \frac{1}{R}(\sin n\theta(\cos^2 \theta + \nu \sin^2 \theta) + n(1-\nu) \sin 2\theta \cos n\theta)(T_n' + \gamma_2 P_n' + \gamma_4 Q_n') - \frac{n}{R^2}(n \sin n\theta(\cos^2 \theta + \nu \sin^2 \theta) + (1-\nu) \sin 2\theta \cos n\theta)(T_n + \gamma_2 P_n + \gamma_4 Q_n) \quad (25d)$$

For free edges at $X = \pm l_1$, Eq. (25) can be applied as follow

$$\sum_{n=0}^{\infty} \{C_n [\Phi_1] + D_n [\Phi_2] + C_n^* [\Phi_3] + D_n^* [\Phi_4]\} = 0, \quad (26)$$

where $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ are shown in the appendix. Eq. (24) holds at $Y = \pm h_2$ clamped edges.

$$\sum_{n=0}^{\infty} \{C_n [Z_1] + D_n [Z_2] + C_n^* [Z_3] + D_n^* [Z_4]\} = 0, \quad (27)$$

where Z_1, Z_2, Z_3, Z_4 are shown in the appendix. The original vector $8N + 4$ unknowns $[A_n, B_n, C_n, D_n, A_n^*, B_n^*, C_n^*, D_n^*]$ has now been reduced to the vector of $4N + 2$ unknowns $[C_n, D_n, C_n^*, D_n^*]$ which now must be determined from Eqs. (21-25). Since the rectangular boundaries are not coordinate lines, an exact solution of Eqs. (21-25) cannot be found. In this section, with the use of a weak solution and also with the use of orthogonality properties of trigonometric functions and multiplying these functions in Eqs. (21-25) and integration of these equations in the intervals of $0 < \theta < 2\pi$, the angular positions is eliminated from the mentioned equations and the equations will be obtained in the form of polynomial functions based one finite degrees of R . It should be mentioned that for the purpose of certainty of the used method, in this section the equations first have been multiplied to $\sin \theta$ and then integrated and then with the use of $\cos \theta$ and due to similarity of the obtained results, finally all the equations have been obtained with the use of orthogonality properties of $\sin \theta$ function. The advantage of using this technique prevents the chosen of many points and reduces the degree of determinant of coefficients matrix. The set of equations which result from satisfying the boundary conditions can be written in matrix form

$$A [C_n, D_n, C_n^*, D_n^*] = 0 \quad (28)$$

where A is a M (Number of boundary conditions) by $4N + 2$ matrix whose coefficients are functions of the frequency ω , and $[C_n, D_n, C_n^*, D_n^*]$ is the $4N + 2$ vector of unknowns. It is shown that a necessary condition that the vectors of unknowns minimize the squared error is that the unknowns satisfy

$$A^T A [C_n, D_n, C_n^*, D_n^*] = 0 \quad (29)$$

where A^T is the transpose of A and $A^T A$ is a $4N + 2$ by $4N + 2$ square matrix. For Eq. (29) to have a nontrivial solution, the determinant of $A^T A$ must vanish. Therefore,

$$|A^T A| = 0 \quad (30)$$

Eq. (30) is the frequency equation which can now be solved to obtain the natural frequency of the plate.

3 NUMERICAL RESULTS

As was stated above, Eq. (30) is the frequency equation which, when solved, will yield the natural frequencies of the plate. Results will now be presented for a square plate, both simply supported and clamped, for a variety of Poisson's ratios. Thus, $a = b$. the infinite series solution (6) is truncated at $n = 4$, leaving eighteen unknowns to be determined. Eight equations for the boundary conditions on the plate can be expressed. Thus Eq. (30) is a determinate equation of order eighteen, and the values of ω which cause the determinate to vanish are the natural frequencies of the plate. In Table 1, the frequency parameter of λ has been shown based on the different radius of the circular hole and different values of ν for a simply supported square plate with a central hole. The boundary conditions along the inner edge are free and along the outer edges are simply supported. Comparisons of the results with reference [28] indicate that the obtained results are having an acceptable accuracy.

As it has been presented in Table 1, with increasing the radius of the hole, the frequency values are first decreased and then increased, this gained a special importance in optimizing the hole radius in analysis these types of problems. In addition, the obtained results indicate that with increasing the value of Poisson's ratio, the frequency values would decrease. In Fig. 2, the first five modes of vibration for simply supported square plate with a central hole have been shown. In Table 2 also the frequency parameter of λ has been shown based on the different radius of the circular hole and different values of ν for a clamped square plate with a central hole and the results are very near to the results of reference [28], which indicate the accuracy of the suggested method. In this section also with increasing the hole radius, the values of frequencies are first decreased and then increased. Only with this difference that when we are using clamped square plate, the values of frequency parameter shows bigger values comparing to the case of simply supported plate.

Table 1
Fundamental natural frequency λ for simply supported square plate with a central hole

$\frac{2r_0}{a}$	ν								
	0.2			0.3			0.4		
a	Present	Ref[28]	FEM	Present	Ref[28]	FEM	Present	Ref[28]	FEM
0.1	19.63	19.61	19.81	19.52	19.53	19.72	19.4	19.3	19.49
0.15	19.59	19.56	19.75	19.38	19.39	19.58	19.13	19.12	19.31
0.2	19.62	19.6	19.79	19.33	19.29	19.48	18.93	18.91	19.09
0.25	19.76	19.72	19.91	19.37	19.32	19.51	18.84	18.8	18.98
0.3	20.11	20.01	20.21	19.62	19.51	19.71	18.94	18.89	19.07

Table 2
Fundamental natural frequency λ for clamped square plate with a central hole

$\frac{2r_0}{a}$	ν								
	0.2			0.3			0.4		
a	Present	Ref[28]	FEM	Present	Ref[28]	FEM	Present	Ref[28]	FEM
0.1	35.29	35.98	35.64	35.11	35.83	35.46	34.92	35.56	35.26
0.15	35.54	36.28	35.89	35.24	35.92	35.59	34.81	35.49	35.16
0.2	36.26	36.98	36.62	35.76	36.41	36.12	35.13	35.87	35.48
0.25	37.36	36.83	37.73	36.75	37.58	37.11	36.02	36.82	36.38
0.3	39.19	38.64	39.02	38.57	39.39	38.95	37.83	38.63	38.21

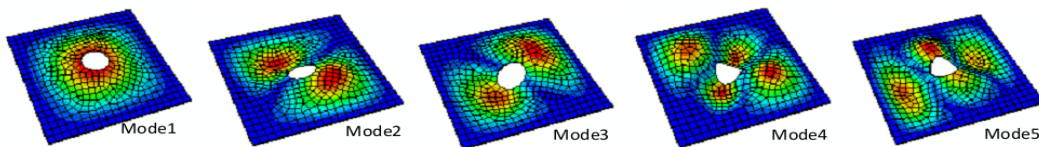


Fig. 2
Modes of vibration for simply supported plate with a circular central hole.

In Fig. 3, the first five modes of vibration for clamped square plate with a central hole have been shown. In Table 3 also the frequency parameter of λ has been shown based on the different radius of the circular hole and different values of ν for a simply supported square plate with a simply supported central hole. In this section also with increasing the hole radius, the values of frequencies are increased. In Fig. 4, the first five modes of vibration for clamped square plate with a simply supported central hole have been shown. In Table 4 also the frequency parameter of λ has been shown based on the different radius of the circular hole and different values of ν for a simply supported square plate with a clamped central hole. In this section also with increasing the hole radius, the values of frequencies are increased. In Fig. 5, the first five modes of vibration for clamped square plate with a central hole have been shown.

Table 3
Fundamental natural frequency λ of a simply supported square plate with a simply supported central hole

$\frac{2r_0}{a}$	ν								
	0.2			0.3			0.4		
a	Present	Ref[28]	FEM	Present	Ref[28]	FEM	Present	Ref[28]	FEM
0.1	52.62	52.65	53.14	50.64	50.75	51.14	48.79	48.9	49.28
0.2	58.95	59.02	59.25	56.73	56.78	57.02	54.67	54.71	54.89
0.3	70.08	70.11	70.64	67.44	67.53	67.98	64.98	65.07	65.52

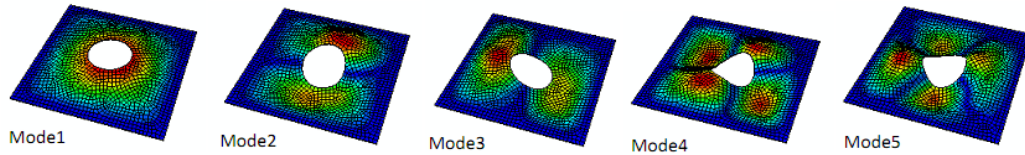


Fig. 3
Modes of vibration for clamped square plate with a circular central hole.

Table 4
Fundamental natural frequency λ of a simply supported square plate with a clamped central hole

$\frac{2r_0}{a}$	ν								
	0.2			0.3			0.4		
a	Present	Ref[28]	FEM	Present	Ref[28]	FEM	Present	Ref[28]	FEM
0.1	62.68	62.79	63.31	60.32	60.42	60.92	58.12	58.22	58.70
0.2	75.92	76.14	76.75	73.12	73.27	73.85	70.46	70.60	71.16
0.3	93.06	93.16	93.99	89.55	89.64	90.44	86.29	86.38	87.15

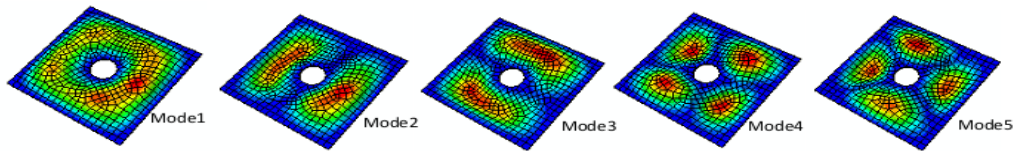


Fig. 4
Modes of vibration for simply supported square plate with a simply supported central hole.

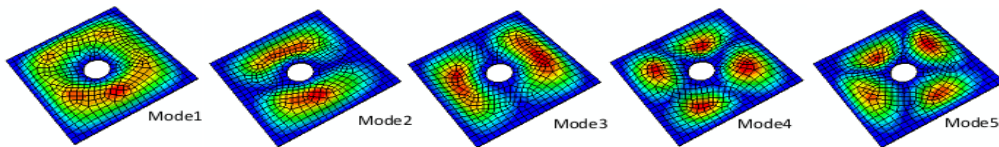


Fig. 5
Modes of vibration for simply supported square plate with a clamped central hole.

In Table 5 also the frequency parameter of λ has been shown based on the different radius of the circular hole and different values of a/b for a simply supported rectangular plate with a circular central hole. In this section also with increasing the hole radius and length of plate, the values of frequencies are increased. In Table 6 also the frequency parameter of λ has been shown based on the different radius of the circular hole and different values of a/b for a simply supported rectangular plate with a circular central hole. In this section also with increasing the hole radius and length of plate, the values of frequencies are increased.

Table 5Fundamental natural frequency λ of a clamped rectangular plate with a circular central hole

$\frac{a}{b}$	$\frac{2r_0}{a}$								
	0.2			0.3			0.4		
	present	Ref[28]	FEM	present	Ref[28]	FEM	present	Ref[28]	FEM
1	35.76	37.83	36.12	38.57	40.63	38.95	45.72	46.35	46.42
1.5	66.1	65.83	65.92	73.42	74.38	74.69	91.88	91.92	91.06
2	110.05	109.79	111.02	127.73	129.46	129.42	160.38	162.97	161.25

Table 6Fundamental natural frequency λ of a simply supported rectangular plate with a circular central hole

$\frac{a}{b}$	$\frac{2r_0}{a}$								
	0.2			0.3			0.4		
	present	Ref[28]	FEM	present	Ref[28]	FEM	present	Ref[28]	FEM
1	19.33	20.24	19.48	19.62	20.95	19.71	20.72	22.15	21.5
1.5	32.23	33.17	33.19	34.6	34.77	34.83	37.64	37.63	37.56
2	51.29	51.12	51.16	53.43	53.67	53.71	57.24	57.98	58.13

4 CONCLUSIONS

In this paper, the free vibration of rectangular plates with circular central hole for various boundary conditions was analyzed and natural frequencies were derived and compared with the reported results of other researchers. To solve the problem, it is necessary that both Cartesian and polar coordinate system be used. For the validation, using the finite element method and modes of vibration for clamped and simply supported square plate with a central hole has been obtained. Comparison of the results obtained from the method used in this article, shows that the results are sufficiently accurate. Also to investigate the problem, Long term and complex relationships, are not used and the problem is simply desired convergence is reached. In this study, the effects of increasing the diameter of the hole on the natural frequencies were investigated and the optimum radius of the circular hole for different boundary conditions are obtained. The optimum value of the radius hole for simply supported square plate at $r_0 = 0.1a$ and in this case will have the least frequency, also the minimum value of the frequency for clamped square plate is $r_0 = 0.075a$.

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APPENDICES

In this section, the coefficients applied to the boundary conditions (26) and (27) are covered.

$$\begin{aligned}
\Phi_1 = & \cos n\theta [\cos^3 \theta + (2-\nu)\cos \theta \sin^2 \theta](S_n''' + \gamma_1 P_n''' + \gamma_3 Q_n''') + \left\{ \frac{n \sin n\theta}{R} [\sin \theta \cos^2 \theta + \sin 2\theta \sin \theta \right. \\
& - (2-\nu)(\cos \theta \sin 2\theta - \sin^3 \theta)] + \frac{\cos n\theta}{R} [3\cos \theta \sin^2 \theta + (2-\nu)(\cos^3 \theta - 2\cos \theta \sin^2 \theta)] \left. \right\} (S_n'' + \gamma_1 P_n'' + \gamma_3 Q_n'') \\
& + \left\{ -\frac{n^2 \cos n\theta}{R^2} [\sin 2\theta \sin \theta + \cos \theta \sin^2 \theta - (2-\nu)(\cos^3 \theta + \sin \theta \sin 2\theta)] - \frac{n \sin n\theta}{R^2} [2\cos 2\theta \sin \theta \right. \\
& + 2\cos \theta \sin 2\theta - \sin^3 \theta - (2-\nu)(2\cos 2\theta \sin \theta + 2\cos \theta \sin 2\theta + \sin \theta \cos^2 \theta)] - \frac{\cos n\theta}{R^2} [\cos \theta \sin^2 \theta \\
& - (2-\nu)(2\cos \theta \sin^2 \theta - \cos^3 \theta)] \left. \right\} (S_n' + \gamma_1 P_n' + \gamma_3 Q_n') + \left\{ -\frac{n^3 \sin n\theta}{R^3} [\sin^3 \theta - (2-\nu)\sin \theta \cos^2 \theta] \right. \\
& + \frac{n^2 \cos n\theta}{R^3} [3\sin \theta \sin 2\theta - (2-\nu)(\sin 2\theta + 2\cos \theta \sin^2 \theta - 2\cos^3 \theta)] + \frac{n \sin n\theta}{R^3} [2\cos 2\theta \sin \theta \\
& + 2\cos \theta \sin 2\theta - (2-\nu)(2\cos 2\theta \sin \theta - 2\cos \theta \sin 2\theta)] \left. \right\} (S_n + \gamma_1 P_n + \gamma_3 Q_n)
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\Phi_2 = & \cos n\theta [\cos^3 \theta + (2-\nu)\cos \theta \sin^2 \theta](T_n''' + \gamma_2 P_n''' + \gamma_4 Q_n''') + \left\{ \frac{n \sin n\theta}{R} [\sin \theta \cos^2 \theta + \sin 2\theta \sin \theta \right. \\
& - (2-\nu)(\cos \theta \sin 2\theta - \sin^3 \theta)] + \frac{\cos n\theta}{R} [3\cos \theta \sin^2 \theta + (2-\nu)(\cos^3 \theta - 2\cos \theta \sin^2 \theta)] \left. \right\} (T_n'' + \gamma_2 P_n'' + \gamma_4 Q_n'') \\
& + \left\{ -\frac{n^2 \cos n\theta}{R^2} [\sin 2\theta \sin \theta + \cos \theta \sin^2 \theta - (2-\nu)(\cos^3 \theta + \sin \theta \sin 2\theta)] - \frac{n \sin n\theta}{R^2} [2\cos 2\theta \sin \theta \right. \\
& + 2\cos \theta \sin 2\theta - \sin^3 \theta - (2-\nu)(2\cos 2\theta \sin \theta + 2\cos \theta \sin 2\theta + \sin \theta \cos^2 \theta)] - \frac{\cos n\theta}{R^2} [\cos \theta \sin^2 \theta \\
& - (2-\nu)(2\cos \theta \sin^2 \theta - \cos^3 \theta)] \left. \right\} (T_n' + \gamma_2 P_n' + \gamma_4 Q_n') + \left\{ -\frac{n^3 \sin n\theta}{R^3} [\sin^3 \theta - (2-\nu)\sin \theta \cos^2 \theta] \right. \\
& + \frac{n^2 \cos n\theta}{R^3} [3\sin \theta \sin 2\theta - (2-\nu)(\sin 2\theta + 2\cos \theta \sin^2 \theta - 2\cos^3 \theta)] + \frac{n \sin n\theta}{R^3} [2\cos 2\theta \sin \theta \\
& + 2\cos \theta \sin 2\theta - (2-\nu)(2\cos 2\theta \sin \theta - 2\cos \theta \sin 2\theta)] \left. \right\} (T_n + \gamma_2 P_n + \gamma_4 Q_n).
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\Phi_3 = & \sin n\theta [\cos^3 \theta + (2-\nu)\cos \theta \sin^2 \theta](S_n''' + \gamma_1 P_n''' + \gamma_3 Q_n''') + \left\{ -\frac{n \cos n\theta}{R} [\sin \theta \cos^2 \theta + \sin 2\theta \sin \theta \right. \\
& - (2-\nu)(\cos \theta \sin 2\theta - \sin^3 \theta)] + \frac{\sin n\theta}{R} [3\cos \theta \sin^2 \theta + (2-\nu)(\cos^3 \theta - 2\cos \theta \sin^2 \theta)] \left. \right\} (S_n'' + \gamma_1 P_n'' + \gamma_3 Q_n'') \\
& + \left\{ -\frac{n^2 \sin n\theta}{R^2} [\sin 2\theta \sin \theta + \cos \theta \sin^2 \theta - (2-\nu)(\cos^3 \theta + \sin \theta \sin 2\theta)] + \frac{n \cos n\theta}{R^2} [2\cos 2\theta \sin \theta \right. \\
& + 2\cos \theta \sin 2\theta - \sin^3 \theta - (2-\nu)(2\cos 2\theta \sin \theta + 2\cos \theta \sin 2\theta + \sin \theta \cos^2 \theta)] - \frac{\sin n\theta}{R^2} [\cos \theta \sin^2 \theta \\
& - (2-\nu)(2\cos \theta \sin^2 \theta - \cos^3 \theta)] \left. \right\} (S_n' + \gamma_1 P_n' + \gamma_3 Q_n') + \left\{ \frac{n^3 \cos n\theta}{R^3} [\sin^3 \theta + (2-\nu)\sin \theta \cos^2 \theta] \right. \\
& + \frac{n^2 \sin n\theta}{R^3} [3\sin \theta \sin 2\theta - (2-\nu)(\sin 2\theta + 2\cos \theta \sin^2 \theta - 2\cos^3 \theta)] - \frac{n \cos n\theta}{R^3} [2\cos 2\theta \sin \theta \\
& + 2\cos \theta \sin 2\theta - (2-\nu)(2\cos 2\theta \sin \theta - 2\cos \theta \sin 2\theta)] \left. \right\} (S_n + \gamma_1 P_n + \gamma_3 Q_n)
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\Phi_4 = & \sin n\theta [\cos^3 \theta + (2-\nu)\cos \theta \sin^2 \theta] (T_n'''' + \gamma_2 P_n'''' + \gamma_4 Q_n''') + \left\{ -\frac{n \cos n\theta}{R} [\sin \theta \cos^2 \theta + \sin 2\theta \sin \theta \right. \\
& - (2-\nu)(\cos \theta \sin 2\theta - \sin^3 \theta)] + \frac{\sin n\theta}{R} [3\cos \theta \sin^2 \theta + (2-\nu)(\cos^3 \theta - 2\cos \theta \sin^2 \theta)] \} (T_n'' + \gamma_2 P_n'' + \gamma_4 Q_n'') \\
& + \left\{ -\frac{n^2 \sin n\theta}{R^2} [\sin 2\theta \sin \theta + \cos \theta \sin^2 \theta - (2-\nu)(\cos^3 \theta + \sin \theta \sin 2\theta)] + \frac{n \cos n\theta}{R^2} [2\cos 2\theta \sin \theta \right. \\
& + 2\cos \theta \sin 2\theta - \sin^3 \theta - (2-\nu)(2\cos 2\theta \sin \theta + 2\cos \theta \sin 2\theta + \sin \theta \cos^2 \theta)] - \frac{\sin n\theta}{R^2} [\cos \theta \sin^2 \theta \\
& - (2-\nu)(2\cos \theta \sin^2 \theta - \cos^3 \theta)] \} (T_n' + \gamma_2 P_n' + \gamma_4 Q_n') + \left\{ \frac{n^3 \cos n\theta}{R^3} [\sin^3 \theta + (2-\nu)\sin \theta \cos^2 \theta] \right. \\
& + \frac{n^2 \sin n\theta}{R^3} [3\sin \theta \sin 2\theta - (2-\nu)(\sin 2\theta + 2\cos \theta \sin^2 \theta - 2\cos^3 \theta)] - \frac{n \cos n\theta}{R^3} [2\cos 2\theta \sin \theta \\
& + 2\cos \theta \sin 2\theta - (2-\nu)(2\cos 2\theta \sin \theta - 2\cos \theta \sin 2\theta)] \} (T_n + \gamma_2 P_n + \gamma_4 Q_n).
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
Z_1 = & \cos n\theta [\sin^3 \theta + (2-\nu)\cos^2 \theta \sin \theta] (S_n'''' + \gamma_1 P_n'''' + \gamma_3 Q_n''') + \left\{ -\frac{n \sin n\theta}{R} [3\sin^2 \theta \cos \theta + (2-\nu) \right. \\
& (\cos^3 \theta - \sin \theta \sin 2\theta)] + \frac{\cos n\theta}{R} [3\sin \theta \cos^2 \theta + (2-\nu)(\sin^3 \theta - 2\cos^2 \theta \sin \theta)] \} (S_n'' + \gamma_1 P_n'' + \gamma_3 Q_n'') \\
& + \left\{ -\frac{n^2 \cos n\theta}{R^2} [3\cos^2 \theta \sin \theta + (2-\nu)(\sin^3 \theta + \cos \theta \sin 2\theta)] - \frac{n \sin n\theta}{R^2} [2\cos 2\theta \cos \theta - \sin \theta \sin 2\theta \right. \\
& + (2-\nu)(\sin \theta \sin 2\theta - 2\cos 2\theta \cos \theta)] + \frac{\cos n\theta}{R^2} [-3\sin \theta \cos^2 \theta + (2-\nu)(2\sin \theta \cos^2 \theta - \sin^3 \theta)] \} \\
& (S_n' + \gamma_1 P_n' + \gamma_3 Q_n') + \left\{ \frac{n^3 \sin n\theta}{R^3} [\cos^3 \theta + (2-\nu)\cos \theta \sin^2 \theta] + \frac{n^2 \cos n\theta}{R^3} [6\sin \theta \cos^2 \theta - (2-\nu) \right. \\
& (4\sin \theta \cos^2 \theta - 2\sin^3 \theta)] - \frac{n \sin n\theta}{R^3} [2\sin 2\theta \sin \theta - 2\cos \theta \cos 2\theta + (2-\nu)(2\cos \theta \cos 2\theta \\
& - 2\sin \theta \sin 2\theta)] \} (S_n + \gamma_1 P_n + \gamma_3 Q_n)
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
Z_2 = & \cos n\theta [\sin^3 \theta + (2-\nu)\cos^2 \theta \sin \theta] (T_n'''' + \gamma_2 P_n'''' + \gamma_4 Q_n''') + \left\{ -\frac{n \sin n\theta}{R} [3\sin^2 \theta \cos \theta + (2-\nu) \right. \\
& (\cos^3 \theta - \sin \theta \sin 2\theta)] + \frac{\cos n\theta}{R} [3\sin \theta \cos^2 \theta + (2-\nu)(\sin^3 \theta - 2\cos^2 \theta \sin \theta)] \} (T_n'' + \gamma_2 P_n'' + \gamma_4 Q_n'') \\
& + \left\{ -\frac{n^2 \cos n\theta}{R^2} [3\cos^2 \theta \sin \theta + (2-\nu)(\sin^3 \theta + \cos \theta \sin 2\theta)] - \frac{n \sin n\theta}{R^2} [2\cos 2\theta \cos \theta - \sin \theta \sin 2\theta \right. \\
& + (2-\nu)(\sin \theta \sin 2\theta - 2\cos 2\theta \cos \theta)] + \frac{\cos n\theta}{R^2} [-3\sin \theta \cos^2 \theta + (2-\nu)(2\sin \theta \cos^2 \theta - \sin^3 \theta)] \} \\
& (T_n' + \gamma_2 P_n' + \gamma_4 Q_n') + \left\{ \frac{n^3 \sin n\theta}{R^3} [\cos^3 \theta + (2-\nu)\cos \theta \sin^2 \theta] + \frac{n^2 \cos n\theta}{R^3} [6\sin \theta \cos^2 \theta - (2-\nu) \right. \\
& (4\sin \theta \cos^2 \theta - 2\sin^3 \theta)] - \frac{n \sin n\theta}{R^3} [2\sin 2\theta \sin \theta - 2\cos \theta \cos 2\theta + (2-\nu)(2\cos \theta \cos 2\theta \\
& - 2\sin \theta \sin 2\theta)] \} (T_n + \gamma_2 P_n + \gamma_4 Q_n).
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
Z_3 = & \sin n\theta [\sin^3 \theta + (2-\nu) \cos^2 \theta \sin \theta] (S_n''' + \gamma_1 P_n''' + \gamma_3 Q_n''') + \left\{ \frac{n \sin n\theta}{R} [3 \sin^2 \theta \cos \theta + (2-\nu) \right. \\
& (\cos^3 \theta - \sin \theta \sin 2\theta)] + \frac{\sin n\theta}{R} [3 \sin \theta \cos^2 \theta + (2-\nu)(\sin^3 \theta - 2 \cos^2 \theta \sin \theta)] \} (S_n'' + \gamma_1 P_n'' + \gamma_3 Q_n'') \\
& + \left\{ -\frac{n^2 \sin n\theta}{R^2} [3 \cos^2 \theta \sin \theta + (2-\nu)(\sin^3 \theta + \cos \theta \sin 2\theta)] + \frac{n \cos n\theta}{R^2} [2 \cos \theta \cos 2\theta - \sin \theta \sin 2\theta \right. \\
& + (2-\nu)(\sin \theta \sin 2\theta - 2 \cos 2\theta \cos \theta)] + \frac{\sin n\theta}{R^2} [-3 \sin \theta \cos^2 \theta + (2-\nu)(2 \sin \theta \cos^2 \theta - \sin^3 \theta)] \} \\
& (S_n' + \gamma_1 P_n' + \gamma_3 Q_n') + \left\{ -\frac{n^3 \cos n\theta}{R^3} [\cos^3 \theta + (2-\nu) \cos \theta \sin^2 \theta] + \frac{n^2 \sin n\theta}{R^3} [6 \sin \theta \cos^2 \theta - (2-\nu) \right. \\
& (4 \sin \theta \cos^2 \theta - 2 \sin^3 \theta)] + \frac{n \cos n\theta}{R^3} [2 \sin \theta \sin 2\theta - 2 \cos \theta \cos 2\theta + (2-\nu)(2 \cos \theta \cos 2\theta \\
& - 2 \sin \theta \sin 2\theta)] \} (S_n + \gamma_1 P_n + \gamma_3 Q_n)
\end{aligned} \tag{A.7}$$

$$\begin{aligned}
Z_4 = & \sin n\theta [\sin^3 \theta + (2-\nu) \cos^2 \theta \sin \theta] (T_n''' + \gamma_2 P_n''' + \gamma_4 Q_n''') + \left\{ \frac{n \sin n\theta}{R} [3 \sin^2 \theta \cos \theta + (2-\nu) \right. \\
& (\cos^3 \theta - \sin \theta \sin 2\theta)] + \frac{\sin n\theta}{R} [3 \sin \theta \cos^2 \theta + (2-\nu)(\sin^3 \theta - 2 \cos^2 \theta \sin \theta)] \} (T_n'' + \gamma_2 P_n'' + \gamma_4 Q_n'') \\
& + \left\{ -\frac{n^2 \sin n\theta}{R^2} [3 \cos^2 \theta \sin \theta + (2-\nu)(\sin^3 \theta + \cos \theta \sin 2\theta)] + \frac{n \cos n\theta}{R^2} [2 \cos \theta \cos 2\theta - \sin \theta \sin 2\theta \right. \\
& + (2-\nu)(\sin \theta \sin 2\theta - 2 \cos 2\theta \cos \theta)] + \frac{\sin n\theta}{R^2} [-3 \sin \theta \cos^2 \theta + (2-\nu)(2 \sin \theta \cos^2 \theta - \sin^3 \theta)] \} \\
& (T_n' + \gamma_2 P_n' + \gamma_4 Q_n') + \left\{ -\frac{n^3 \cos n\theta}{R^3} [\cos^3 \theta + (2-\nu) \cos \theta \sin^2 \theta] + \frac{n^2 \sin n\theta}{R^3} [6 \sin \theta \cos^2 \theta - (2-\nu) \right. \\
& (4 \sin \theta \cos^2 \theta - 2 \sin^3 \theta)] + \frac{n \cos n\theta}{R^3} [2 \sin \theta \sin 2\theta - 2 \cos \theta \cos 2\theta + (2-\nu)(2 \cos \theta \cos 2\theta \\
& - 2 \sin \theta \sin 2\theta)] \} (T_n + \gamma_2 P_n + \gamma_4 Q_n).
\end{aligned} \tag{A.8}$$

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