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On the commuting graph of some non-commutative rings with unity

E. Vatandoost[a], F. Ramezani[a∗]

[a]*Department of Basic science , Imam Khomeini International University, Qazvin, Iran.*

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Abstract. Let R be a non-commutative ring with unity. The commuting graph of R denoted by $\Gamma(R)$, is a graph with a vertex set $R \setminus Z(R)$ and two vertices a and b are adjacent if and only if $ab = ba$. In this paper, we investigate non-commutative rings with unity of order p^n where p is prime and $n \in \{4, 5\}$. It is shown that, $\Gamma(R)$ is the disjoint union of complete graphs. Finally, we prove that there are exactly five commuting graphs of non-commutative rings with unity up to twenty vertices and they are $3K_2, 3K_4, 7K_2, K_2 \cup 2K_6$ and $4K_2 \cup K_6$.

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1. Introduction

The study of algebraic structures has become an exciting research topic in recent years. One of the algebraic graphs is commuting graph which was introduced in [2]. Let R be a non-commutative ring with a unity 1 and let $Z(R)$ denote the center of R. We assume $1 \neq 0$. A ring with a unity is a division ring if every non-zero element a has a multiplicative inverse (that is, an element x with $ax = xa = 1$). If X is either an element or a subset of the ring R, then $C_R(X)$ denote the *centralizer* of X in R. We introduce a graph with the vertex set $R \setminus Z(R)$ and join two vertices a and b if $a \neq b$ and $ab = ba$. This graph is called a *commuting graph* of R and is denoted by $\Gamma(R)$. Akbari et.al [3] determined the diameters of some induced subgraphs of $\Gamma(M_n(D))$, for a division ring

∗Corresponding author.
E-mail address: ramezani@ikiu.ac.ir (F. Ramezani).

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D and $n \geqslant 3$. Also they showed that if F is an algebraically closed field or n is a prime number and $\Gamma(M_n(F))$ is a connected graph, then diameter of $\Gamma(M_n(F))$ is equal to 4.

Let G be a simple graph on a vertex set $V(G)$ and edge set $E(G)$. A graph is said to be *connected* if each pair of vertices are joined by a walk. If G is a graph, then the *complement* of G, denoted by G^c is a graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G. The *complete graph* K_n is the graph with n vertices in which each pair of vertices are adjacent. We show $G = tK_m$ for disjoint union of t complete graph of size m. G is *complete* $t-$*partite* graph if there is a partition $V_1 \cup V_2 \cup \ldots \cup V_t = V(G)$ of the vertex set, such that v_i and v_j are adjacent if and only if v_i and v_j are in different parts of the partition. If $|V_k| = n_k$, then G is denoted by $K_{n_1,n_2,\ldots,n_t}$.

In this paper, we investigate a non-commutative ring with a unity of order p^n where p is prime and $n \in \{4, 5\}$. We determine that $C_R(a)$ is a commutative ring for every $a \in R \setminus Z(R)$. In addition, it is shown that, $\Gamma(R)$ is the disjoint union of complete graphs. Furthermore we prove that a graph with p^n vertices where $n < 4$ is not a commuting graph of a non-commutative ring with a unity. Finally, we show that there are exactly five commuting graphs of non-commutative rings with a unity up to twenty vertices and they are $3K_2, 3K_4, 7K_2, K_2 \cup 2K_6$ and $4K_2 \cup K_6$.

1.1 *Preliminaries*

First we give some results that we will use them in the next section.

Lemma 1.1 [2] Let R be a non-commutative ring and $a - b$ be an edge in $\Gamma(R)^c$. Then there is a triangle $a - (a + b) - b - a$ containing the edge $a - b$ in $\Gamma(R)^c$.

Lemma 1.2 [2] For any non-commutative ring R and $x, y \in V(\Gamma(R)^c)$, there is a path between x and y in $\Gamma(R)^c$ whose length is at most two.

Lemma 1.3 [7] Let R be a non-commutative ring with unity. Then $[R : Z(R)] \geqslant 4$.

Lemma 1.4 [8] Let R be a non-commutative ring and $Z(R) \neq \{0\}$. Then $[R : Z(R)]$ is not prime.

Lemma 1.5 [6] Let R be a finite ring of order p^n with unity, where p is a prime. If $n < 3$, then R is commutative.

Lemma 1.6 [6] Let R be a finite ring of order m with a unity. If m has a cube free factorization, then R is a commutative ring.

Lemma 1.7 [8] Let R be a non-commutative ring with unity and $|R| = p^3$, then $|Z(R)| = p$.

Lemma 1.8 Let R be a finite non-commutative ring with unity. Then

$$|Z(R)| \mid |R \setminus Z(R)|.$$

Proof. The proof is straightforward. ■

Theorem 1.9 Let R be a non-commutative ring with unity. Then $\Gamma(R)$ is a finite graph if and only if R is a finite ring.

Proof. Let $\Gamma(R)$ be a graph of ordet m. Then $|R \setminus Z(R)| = m$ and so $[R : Z(R)] = t < \infty$. If $R = Z(R) \cup (a_1 + Z(R)) \cup \ldots \cup (a_{t-1} + Z(R))$, then $|(a_1 + Z(R)) \cup \ldots \cup (a_{t-1} + Z(R))| = m$. Thus $|R| = |Z(R)| + m$ and $|Z(R)| < m$. The converse is clear. ■

Lemma 1.10 Let R be a non-commutative ring with unity and $[R : Z(R)] \leqslant 7$. Then $\Gamma(R)$ is not a connected graph.

Proof. By Lemma 1.4, $[R : Z(R)]$ is not prime. By Lemma 1.3, $[R : Z(R)] \in \{4, 6\}$. If $[R : Z(R)] = 4$, then $R = Z(R) \cup (a + Z(R)) \cup (b + Z(R)) \cup (c + Z(R))$ where $a, b, c \in R \setminus Z(R)$. If $ab = ba$, then $R = C_R(a) \cup C_R(c)$. Hence $C_R(a) \subseteq C_R(c)$ or $C_R(c) \subseteq C_R(a)$. This is contradiction by R is not commutative. Let $|Z(R)| = t$. Then $\Gamma(R)^c = K_{t,t,t}$.

If $[R : Z(R)] = 6$, then $R = Z(R) \cup (a_1 + Z(R)) \cup ... \cup (a_5 + Z(R))$ where $a_i \in R \setminus Z(R)$ for $1 \leqslant i \leqslant 5$. By Lemmas 1.1 and 1.2, there exists $1 \leqslant k \leqslant 5$ such that every elements of $a_k + Z(R)$ are adjacent to every element of $a_i + Z(R)$ for $i \in \{1, \ldots, 5\} \setminus \{k\}$ as vertices in $\Gamma(R)^c$. Since induced subgraph of $a_k + Z(R)$ in $\Gamma(R)$ is a complete graph of size t, K_t is one of the components of $\Gamma(R)$. Therefore $\Gamma(R)$ is not a connected graph. ■

2. Commuting graph of non-commutative rings with unity of order p^n

In this section, we consider the commuting graph of non-commutative rings with unity of order p^n where p is prime and $n \in \{4, 5\}$.

Theorem 2.1 Let R be a non-commutative ring with a unity of order p^4 and $a \in R \setminus Z(R)$. Then $C_R(a)$ is a commutative ring.

Proof. We know that $|Z(R)| \in \{1, p, p^2, p^3\}$. Since R is a non-commutative ring with unity, $|Z(R)| = p$ or p^2.

Let $|Z(R)| = p$. Since $C_R(a)$ is an addition subgroup of R and $a \notin Z(R)$, $|C(R)| = p^2$ or p^3. If $|C_R(a)| = p^2$, then by Lemma 1.5, $C_R(a)$ is a commutative ring. Suppose that $|C_R(a)| = p^3$ and $C_R(a)$ be a non-commutative ring. By Lemma 1.7 , $|Z(C_R(a))| = p$. It is clear that $Z(R) \cup (a + Z(R)) \subseteq Z(C_R(a))$. Thus $2p \leqslant p$. It is impossible. Therefore $C_R(a)$ is a commutative ring.

Let $|Z(R)| = p^2$. Then $|C_R(a)| = p^3$. If $C_R(a)$ is a non-commutative ring, then by Lemma 1.7, $|Z(C_R(a))| = p$. But $Z(R) \subseteq Z(C_R(a))$. This is not true and so $C_R(a)$ is a commutative ring. ■

Lemma 2.2 Let R be a non-commutative ring with unity of order p^4. If $a, b \in R \setminus Z(R)$ and $ab = ba$, then $C_R(a) = C_R(b)$.

Proof. Let $x \in C_R(a)$. By Theorem 2.1, $xb = bx$ and so $x \in C_R(a)$. Thus $C_R(a) \subseteq C_R(b)$. Similarly $C_R(b) \subseteq C_R(a)$. Therefore $C_R(a) = C_R(b)$. ■

Theorem 2.3 Let R be a non-commutative ring with a unity of order p^4. If $a, b \in R \setminus Z(R)$ and $ab \neq ba$, then $C_R(a) \cap C_R(b) = Z(R)$.

Proof. If there exists a $x \in C_R(a) \cap C_R(b) \setminus Z(R)$, then by Lemma 2.2, $C_R(x) = C_R(a)$ and $C_R(x) = C_R(b)$. Thus $C_R(a) = C_R(b)$ and so $ab = ba$, a contradiction. ■

Lemma 2.4 Let R be a non-commutative ring with unity of order p^4 and $|Z(R)| = p$. Then there exist $a \in R \setminus Z(R)$ such that $|C_R(a)| = p^2$.

Proof. Since R is a non-commutative ring, $|C_R(a)| = p^2$ or p^3 for $a \in R \setminus Z(R)$. On the contrary, suppose $|C_R(a)| = p^3$ for every $a \in R \setminus Z(R)$. Let $a, b \in R \setminus Z(R)$ and $ab \neq ba$. If there exists $x \in C_R(a), y \in C_R(b)$ such that $xy = yx$, then by Lemma 2.2, $C_R(a) = C_R(x)$, $C_R(b) = C_R(y)$ and $C_R(x) = C_R(y)$. So $C_R(a) = C_R(b)$. This is not true. So for every $x \in C_R(a)$ and $b \in C_R(b)$, $xy \neq yx$. By Theorem 2.3, $C_R(a) \cap C_R(b) = Z(R)$. Thus $\Gamma(R)$ is the disjoint union of l copies of the complete graph K_{p^3-p}. So $|V(\Gamma(R))| =$

$l(p^3 - p)$. On the other hand, we have $|V(\Gamma(R)| = |R| - |Z(R)| = p^4 - p$. Therefore $p^4 - p = l(p^3 - p)$ and so $p^2 + p + 1 = l(p + 1)$ which is not true. ∎

Theorem 2.5 Let R be a non-commutative ring with a unity of order p^4. Then the commuting graph of R is one of the following cases:

i. $\Gamma(R) = (p^2 + p + 1)K_{(p^2-p)}$.
ii. $\Gamma(R) = l_1K_{(p^2-p)} \bigcup l_2K_{(p^3-p)}$, where $l_1 + l_2(p + 1) = p^2 + p + 1$.
iii. $\Gamma(R) = (p + 1)K_{(p^3-p^2)}$.

Proof. It follows immediately that $|Z(R)| = p$ or p^2. So the proof will be divided into two cases:

Case 1. Let $|Z(R)| = p$. By Lemma 2.4, there is $a \in R \setminus Z(R)$ such that $|C_R(a)| = p^2$. Suppose that $|C_R(a)| = p^2$ for every $a \in R\setminus Z(R)$. Let $a, b \in R\setminus Z(R)$ and $ab \neq ba$. By Theorem 2.3, $C_R(a) \cap C_R(b) = Z(R)$. If $x \in C_R(a), y \in C_R(b)$ and $xy = yx$, then by Theorem 2.3, $C_R(a) = C_R(x)$, $C_R(b) = C_R(y)$ and $C_R(x) = C_R(y)$. So $C_R(a) = C_R(b)$, which is impossible. Therefore, $\Gamma(R)$ is the disjoint union of l copies the complete graph K_{p^2-p}. So $|V(\Gamma(R))| = l(p^2 - p)$. On the other hand, we have $|V(\Gamma(R))| = |R| - |Z(R)| = p^4 - p$. Thus $p^4 - p = l(p^2 - p)$ and as consequence $l = p^2 + p + 1$, and (i) is proved.
Let $a, b \in R \setminus Z(R)$, $|C_R(a)| = p^2$ and $|C_R(b)| = p^3$. By Theorem 2.3, $C_R(a) \cap C_R(b) = Z(R)$. It is easy to check that if $x \in C_R(a)$ and $y \in C_R(b)$, then $xy \neq yx$. Hence $\Gamma(R)$ is the disjoint union of l_1 copies of the complete graph K_{p^2-p} and l_2 copies of the complete graph K_{p^3-p}. So $|V(\Gamma(R))| = l_1(p^2 - p) + l_2(p^3 - p)$. On the other hand, we have $|V(\Gamma(R))| = p^4 - p$. Thus $p^4 - p = l_1(p^2 - p) + l_2(p^3 - p)$. Therefore $\Gamma(R) = l_1K_{(p^2-p)} \cup l_2K_{(p^3-p)}$, where l_1 and l_2 satisfy in $l_1 + l_2(p+1) = p^2 + p + 1$, and part (ii) is proved.

Case 2. Let $|Z(R)| = p^2$. Then $|C_R(x)| = p^3$ for every $x \in R \setminus Z(R)$. Suppose that $a, b \in R \setminus Z(R)$ and $ab \neq ba$. By Theorem 2.3, $C_R(a) \cap C_R(b) = Z(R)$. Also if $x \in C_R(a)$ and $y \in C_R(b)$, then $xy \neq yx$. Thus $\Gamma(R)$ is the disjoint union of l copies of the complete graph of size $p^3 - p^2$ and so $|V(\Gamma(R))| = l(p^3 - p^2)$. Since $|V(\Gamma(R))| = p^4 - p^2$, $p^4 - p^2 = l(p^3 - p^2)$. Therefore $\Gamma(R) = lK_{(p^3-p^2)}$ where $l = p + 1$, and this completes the proof of (iii).

∎

Lemma 2.6 Let R be a non-commutative ring with a unity of order p^5 such that $Z(R)$ is not a field. Then the following is hold:

i. For every $a \in R \setminus Z(R)$, $C_R(a)$ is a commutative ring.
ii. If $a, b \in R \setminus Z(R)$ such that $ab = ba$, then $C_R(a) = C_R(b)$.
iii. If $a, b \in R \setminus Z(R)$ such that $ab \neq ba$, then $C_R(a) \cap C_R(b) = Z(R)$.

Proof. It is not hard to see that $|Z(R)|$ is p^2 or p^3. Since $Z(R)$ is an addition subgroup of $C_R(a)$ and R is not commutative ring, $|C_R(a)| \in \{p^3, p^4\}$. Let $C_R(a)$ be a non-commutative ring of order p^3. Then $|Z(C_R(a))| = p$. This is not true since $Z(R) \subseteq Z(C_R(a))$. If $C_R(a)$ is a non-commutative ring of order p^4, then $|Z(C_R(a))|$ is p or p^2. Since $a \in R\setminus Z(R)$ and $Z(R) \subseteq Z(C_R(a))$, this is impossible. Hence $C_R(a)$ is a commutative ring. The proof of parts (ii) and (iii) are likewise Lemma 2.2 and Theorem 2.3 , respectively. ∎

Theorem 2.7 Let R be a non-commutative ring with unity of order p^5 such that $Z(R)$ is not a field. Then the commuting graph of R is one of the following cases:

i. $\Gamma(R) = (p^2 + p + 1)K_{p^3-p^2}$.
ii. $\Gamma(R) = l_1 K_{p^3-p^2} \cup l_2 K_{p^3-p}$, where $l_1 + l_2(p+1) = p^2 + p + 1$.
iii. $\Gamma(R) = (p+1)K_{p^4-p^3}$.

Proof. Since R is a non-commutative ring and $Z(R)$ is not a field, $|Z(R)| \in \{p^2, p^3\}$.

Case 1. Let $|Z(R)| = p^2$. Then for $a \in R \setminus Z(R)$, $|C_R(a)| = p^3$ or p^4. Suppose that for every $a \in R \setminus Z(R)$, $|C_R(a)| = p^4$. By a similar argument as in Theorem 2.5, if $x, y \in R$, then $xy \neq yx$. Thus $\Gamma(R)$ is the disjoint union of l copies of complete graph $K_{p^4-p^2}$. Since $|V(\Gamma(R))| = p^5 - p$, $p^2 + p + 1 = l(p+1)$. This is not true. So there exists a $b \in R \setminus Z(R)$ such that $|C_R(b)| = p^3$. If for every $a \in R \setminus Z(R)$, $|C_R(a)| = p^3$, then $\Gamma(R) = lK_{p^3-p^2}$ where $l = p^2 + p + 1$. Otherwise, suppose that $|\{a\,;\, |C_R(a)| = p^3\}| = l_1$ and $|\{b\,;\, |C_R(b)| = p^4\}| = l_2$. Thus $\Gamma(R) = l_1 K_{p^3-p^2} \cup l_2 K_{p^4-p^2}$ where $l_1 + l_2(p+1) = p^2 + p + 1$.

Case 2. Let $|Z(R)| = p^3$. Since $Z(R) \subseteq C_R(a)$ for every $a \in R \setminus Z(R)$, $|C_R(a)| = p^4$. Thus $\Gamma(R) = lK_{p^4-p^3}$ where $l = p + 1$. This completes the proof.

■

3. Determine the Commuting graph up to twenty vertices

In this section we show that there are exactly five commuting graphs on non-commutative ring with unity up to twenty vertices.

Lemma 3.1 Let G be a graph with $2p$ vertices where p is an odd prime number and $|V(G)| \neq 6$, then G is not a commuting graph of a non-commutative ring with unity.

Proof. Suppose $G = \Gamma(R)$ where R is a non-commutative ring with a unity. So $|R \setminus Z(R)| = 2p$. By Lemma 1.8, $|Z(R)| \in \{2, p, 2p\}$.

If $|Z(R)| = 2$ or p, then $|R| = 2 + 2p$ or $3p$ respectively. Since $p \neq 3$ by Lemma 1.6, R is a commutative ring. This is contradiction.

If $|Z(R)| = 2p$, then $[R : Z(R)] = 2$. This is a contradiction by Lemma 1.3. ■

Theorem 3.2 Let G be a graph with pq vertices where p and q are two distinct prime numbers and $p < q, p \nmid q + 1$. Then G is not a commuting graph of a non-commutative ring with unity.

Proof. Let R be a non-commutative ring with unity and $G = \Gamma(R)$. We look for a contradiction. $|R \setminus Z(R)| = |V(G)| = pq$. By Lemma 1.8, $|Z(R)| \mid pq$ and so $|Z(R)| \in \{p, q, pq\}$.

If $|Z(R)| = p$, then $|R| = p(q+1)$. Since $p \nmid q + 1$, by Lemma 1.6, R is a commutative ring, which is impossible.

If$|Z(R)| = q$, then $|R| = q(p+1)$. Since $p < q$, by Lemma 1.6, R is a commutative ring. Which is not true.

If $|Z(R)| = pq$, then $[R : Z(R)] = 2$, which is a contradiction by Lemma 1.3. This completes the proof. ■

Lemma 3.3 If G is a graph with p^n vertices where $n < 4$, then G is not a commuting graph of a non-commutative ring with unity.

Proof. On the contrary suppose R is a non-commutative ring with a unity and $G = \Gamma(R)$. Since R has unity, $|Z(R)| \geqslant 2$.

If $|V(G)| = p$, then by Lemma 1.8, $|Z(R)| = p$. Therefore $|Z(R)| = |R \setminus Z(R)|$. So $[R : Z(R)] = 2$ which is a contradiction by Lemma 1.3.

If G has p^2 vertices, then $|Z(R)| \in \{p, p^2\}$. If $|Z(R)| = p^2$, then $|Z(R)| = |R \setminus Z(R)|$ and so $[R : Z(R)] = 2$. This is impossible. Hence $|Z(R)| = p$. So $|R| = p(p+1)$. By Lemma 1.6, R is a commutative ring. This is not true.

Let $|V(G)| = p^3$. By Lemma 1.8, $|Z(R)| \in \{p, p^2, p^3\}$. If $|Z(R)| = p^3$, then $[R : Z(R)] = 2$. This is impossible. Therefore $|R| = p(p^2+1)$ or $p^2(p+1)$. Since $p^2 \nmid (p^2+1)$ and $p \nmid (p+1)$, by Lemma 1.6, R is a commutative ring. This is a contradiction. ■

Theorem 3.4 There are exactly five commuting graphs on non-commutative ring with unity up to twenty vertices. They are $3K_2, 3K_4, 7K_2, K_2 \cup 2K_6$ and $4K_2 \cup K_6$.

Proof. Let G be a commuting graph of a non-commutative ring R with a unity. Let $|V(G)| \leqslant 20$. By Lemmas 3.1, 3.2, 3.3, $|V(G)| \in \{6, 12, 14, 16, 18, 20\}$.

Let $|V(G)| = n$ be even and let $|Z(R)| = \frac{n}{2}$. Then $|R| = \frac{3n}{2}$. So $[R : Z(R)] = 3$. Which is not true. So $|Z(R)| \neq \frac{n}{2}$.

Let $|V(G)| = 16$. By Lemma 1.8, $|Z(R)| \in \{2, 4\}$. So $|R| \in \{18, 20\}$. Hence by Lemma 1.6, R is a commutative ring. This is impossible.

Let $|V(G)| = 18$. Thus $|Z(R)| \in \{2, 3, 6\}$. So $|R| \in \{20, 21, 24\}$. By Lemma 1.6, R is a commutative ring and this is contradiction.

If $|V(G)| = 20$, then by Lemma 1.8, $|Z(R)| \in \{2, 4, 5\}$. Hence $|R| \in \{22, 24, 25\}$. Again R is a commutartive ring.

Therefore $|V(G)| \in \{6, 12, 14\}$. If $|V(G)| = 6$, then by Lemmas 1.3 and 1.8, $|Z(R)| = 2$ and so $|R| = 8$. By the proof of Lemma 1.10, $G = 3K_2$.

If G has 12 vertices, then $|Z(R)| = 4$. So $|R| = 16$. By Theorem 2.5, $G = 3K_4$. Let G be a commuting greaph of order 14. Then $|Z(R)| = 2$ and $|R| = 16$. By Theorem 2.5, $G = 7K_2, K_2 \cup 2K_6$ and $4K_2 \cup K_6$. This completes the proof. ■

Conjecture:

The commuting graph of non-commutative rings with unity of order p^n is not a connected graph.

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