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A New Approach for Solving Interval Quadratic Programming Problem

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Abstract

This paper discusses an Interval Quadratic Programming (IQP) problem, where the constraints coefficients and the right-hand sides are represented by interval data. First, the focus is on a common method for solving Interval Linear Programming problem. Then the idea is extended to the IQP problem. Based on this method each IQP problem is reduced to two classical Quadratic Programming (QP) problems. Afterwards these classical problems are solved using the SQP algorithm and the numerical results are presented.

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INTRODUCTION

QP is a special type of mathematical optimization problem that has been applied to solve real world problems (Jafari, 2010). Since, this subject was studied by many researchers where some of them are as follows. In 2014, Kochenberger et al. studied the unconstrained binary quadratic programming problem (Kochenberger, et al., 2014). Frasch et al. in 2015 addressed the ubiquitous case where the quadratic programming problems are strictly convex and proposed a dual Newton strategy that exploits the block-bandedness similarly to an interior-point method (Frasch, Sager, & Diehl, 2015). At this year, Gill and Wong studied active-set method for a generic quadratic programming problem with both equality and inequality constraints (Gill & Wong, 2015). In 2017, Takapoui et al. proposed a fast optimization algorithm for approximately minimizing convex quadratic functions over the intersection of affine and separable constraints (Takapoui, Moehle, Boyd, & Bemporad, 2017).

In conventional quadratic programming model, the parameters are known constant. However, in the real world, ambiguity and vagueness are natural and ever-present, so the parameters are seldom known exactly. Hence, we use interval programming to model these ambiguities and uncertainties in mathematical forms. Because interval programming problems in comparison with fuzzy programming problems and probabilistic programming problems need less information. Also none of the interior numbers of interval has qualitative and insufficiency preference to other ones. Hence, it is so applicable and effective to use from interval programming in such situations. In interval linear programming problems is of interests of many researchers. As instance some of these studies are taken bellow.

In 1994 Shaocheng (Shaocheng, 1994) studied interval number linear programming problems and proposed a new method based on the maximum value range and minimum value range. Lodwick (Lodwick, 2011), studied explores the interconnections between interval analysis, fuzzy interval analysis, and interval, fuzzy, and possibility optimization. Hladík (Haldik, 2014) proposed a method for testing basis stability of interval linear programming. Hladík (Haldik, 2017) in 2017, dealt with a linear programming problem with interval data and discussed the problem of checking whether a given solution is optimal for each realization of interval data. Allahdadi et al. (Allahdadi, Mishmast Nehi, Ashayerinasab, & Javanmard, 2016) considered the interval linear programming problems, which are used to deal with uncertainties resulting from the range of admissible values in problem coefficients. Then, they proposed two new ILP methods and their sub-models to solve them.

Again at this year Hlad´ık, introduced a novel kind of robustness in linear programming and proposed a method to check for robustness of a given point. He also recommends how a suitable candidate can be found and discussed topological properties of the robust optimal solution set, too. Ashayerinasab et al. (Ashayerinasab, Mishmast-Nehi, & M., 2018) introduced a new algorithm to solve an arbitrary characteristic model of the interval linear programming model. Also, Mishmast et al. (Mishmast Nehi, Ashayerinasab, & Allahdadi, 2018) reviewed some existing methods for solving interval linear programming problems and introduced an improved method and its sub-models in 2018.

Interval programming problems in comparison with fuzzy programming problems and probabilistic programming problems (Ebrahimnejad, Ghomi & Mirhosseini-Alizamini, 2018; Taleshian & Fathali, 2016; Taleshian, Fathali, & Taghi-Nezhad, 2018; Nasseri, Taghi-Nezhad, & Ebrahimnejad, 2017a; Nasseri, Taghi-Nezhad, & Ebrahimnejad, 2017b; Khalili Goodarzi, Taghinezhad, & Nasseri, 2014) need to much less information (Rezai Balf, Hosseinzadeh Lotfi, & Alizadeh Afrouzi, 2010). Also none of the interior numbers of interval has qualitative and insufficiency preference to other ones. Hence, it is so applicable and effective to use from interval programming in such situations.

In spite of the fact that interval linear programming problem was noticed by many researchers as an interesting subject, no much observable development is happened in interval quadratic programming problem. Hence in this paper we deal with interval quadratic programming problem.

This paper is organized in 6 sections. In the next section, quadratic programming problem is defined. In Section 3 some necessary notations and definitions of interval numbers and interval arithmetic are given. Section 4 with its three subsections, defines an IQP problem for the quadratic form and extends a pre-existing method which was applied for the linear form (Shocheng, 1994). In Section 5, an example is presented to illustrate how to apply the contribution of this paper for solving the QP problem with interval parameters. Finally, a conclusion is drawn and some directions for future study is suggested in Section 6.

QUADRATIC PROGRAMMING

QP problem is the problem of optimizing (minimizing or maximizing) a quadratic function of several variables subject to linear constraints on these variables and can be formulated as follows:

$$\text{Min } Z = \sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} x_i x_j \tag{1}$$
$$S.t, \begin{cases} \sum_{j=1}^{n} a_{ij} x_j \geq b_i, & i = 1,2,\dots,m \\ x_j \geq 0, & j = 1,2,\dots n \end{cases}$$

In many of engineering applications including regression analysis, image and signal progressing, parameter estimation, filter design, robust control, and so on (Aboudolas, Papageorgiou, Kouvelas, & Kosmatopoulos, 2010; Gupta, 1995; Hertoga, Roosa, & Terlakya, 1991; Zhou, Cheng, & Li, 2012; Taghi-Nezhad & Taleshian, 2018), it is necessary to solve the quadratic programming problem (1). The problem in vector-matrix notation may be written as follows:

$$\text{Min } z = cx + \frac{1}{2}x^T Q x \tag{2}$$
$$S.t, \begin{cases} A\boldsymbol{x} \geq b \\ \boldsymbol{x} \geq 0 \end{cases}$$

where $x=(x_1, x_2, \dots, x_n)^T$ is the vector of decision variables which should be determined. The others are the parameters given by problem: $c=(c_1,c_2,\dots,c_n)$ is the vector of cost coefficients, $Q=[q_{ij}]_{m\times n}$ is the matrix of quadratic form, $A=[a_{ij}]_{m\times n}$ is the matrix of constraint coefficients and $b=(b_1,b_2,\dots,b_m)^T$ is the vector of right-hand side. Any QP over linear constraints can be put in the above standard form. Without loss of generality, we assume that Q is symmetric and positive semi-definite. Thus, QP is solvable in polynomial time (Kozlov et al., 1980). Although, it is available to consider Q as an indefinite or negative semi-definite matrix, in which case QP is an NP-hard problem (Pardalos & Vavasis, 1991). As it mentioned before, QP models are usually formulated to find some future courses of action. Hence the parameter values used would be on a prediction of future conditions which inevitably involves some uncertainty. In such cases the uncertainties can be modeled in mathematical forms using interval numbers (Ishizaki, et al., 2016; Lio & Wang, 2007; Wang & Huang, 2013). In this paper, the programming problems are dealt with where the parameters are represented by interval data. Since the parameters are interval valued, the objective value is interval valued as well.

THE BASIC INTERVAL ARITHMETIC

Some basic definitions and properties of interval numbers and intervals arithmetic can be seen in (Mohd, 2006).

Definition 1: An interval number is a closed interval such as

$$\tilde{A}=[a_L, a_R]=\{a: a_L\leq a \leq a_R, a\in\mathbb{R}\}$$

where a_L and a_R are left and right limit of interval $\tilde{A}$ on the real line $\mathbb{R}$ respectively. If $a_L=a_R$, then $\tilde{A}=[a,a]$ is a real number.

Definition 2: Any interval number $\tilde{A}$ is alternatively represented as $\tilde{A}=(m(\tilde{A}),w(\tilde{A}))$, where $m(\tilde{A})$ and $w(\tilde{A})$ are called the mid-point and half-width of interval $\tilde{A}$, and are defined as follows:

$$m(\tilde{A})=1/2\ (a_L+a_R),\quad w(\tilde{A})=1/2\ (a_R - a_L) \tag{3}$$

Definition 3: Let [a,b] and [c,d] be two arbitrary interval numbers. Therefore, four arithmetic operations [addition +, subtraction -, multiplication ×, and division /] on interval numbers are defined as follows:

$$[a,b]+[c,d]=[a+c,b+d] \tag{4}$$
$$[a,b]-[c,d]=[a-d,b-c] \tag{5}$$

$$[a,b]\times[c,d]=[\min\{ac, ad, bc, bd\},\max\{ac, ad, bc, bd\}] \tag{6}$$

$$[a,b]/[c,d]= [a,b]\times[1/d,1/c][\min\{a/c,a/d,b/c,b/d\}, \max\{a/c,a/d,b/c,b/d\}],\quad 0\in[c,d] \tag{7}$$

Arithmetic operations on interval numbers satisfy useful properties. To give an overview of them, let $\tilde{A}=[a_L, a_R]$, $\tilde{B}=[b_L,b_R]$, $\tilde{C}=[c_L,c_R]$, $0=[0,0]$ and $1=[1,1]$. Using these symbols, the properties are formulated as follows [2]:

1. $\tilde{A}+\tilde{B}=\tilde{B}+\tilde{A}$
   $\tilde{A}\times\tilde{B}=\tilde{B}\times\tilde{A}$, (commutativity)
2. $(\tilde{A}+\tilde{B})+\tilde{C}=\tilde{A}+(\tilde{B}+\tilde{C})$
   $(\tilde{A}\times\tilde{B})\times\tilde{C}=\tilde{A}\times(\tilde{B}\times\tilde{C})$, (associativity)
3. $\tilde{A}=\tilde{A}+0=0+\tilde{A}$
   $\tilde{A}=\tilde{A}\times 1=1\times\tilde{A}$, (identity)
4. $\tilde{A}\times(\tilde{B}+\tilde{C})\subseteq\tilde{A}\times\tilde{B}+\tilde{A}\times\tilde{C}$, (subdistributivity)
5. If $b\times c\geq 0$, for any $b\in\tilde{B}$ and $c\in\tilde{C}$, then $\tilde{A}\times(\tilde{B}+\tilde{C})=\tilde{A}\times\tilde{B}+\tilde{A}\times\tilde{C}$.

Furthermore, if $\tilde{A}=[a,a]$, then $\tilde{A}\times(\tilde{B}+\tilde{C})=a\times\tilde{B}+a\times\tilde{C}$, (distributivity)

6. $0\in\tilde{A}-\tilde{A}$ and $1\in\tilde{A}/\tilde{A}$.
7. If $\tilde{B}\subseteq E$ and $\tilde{C}\subseteq F$, then
   $\tilde{B}+\tilde{C}\subseteq E+F$,
   $\tilde{B}-\tilde{C}\subseteq E-F$,
   $\tilde{B}\cdot\tilde{C}\subseteq E\cdot F$,
   $\tilde{B}/\tilde{C}\subseteq E/F$, (inclusion monotonicity).

INTERVAL QUADRATIC PROGRAMMING

Here, we first define IQP models and then propose a novel method for solving the mentioned problem.

Definition of IQP model

If the parameters cannot be exactly known constant and presented by interval data the QP problem transforms to the IQP problem and in general form can be written as follows:

$$\text{Min } Z=\sum_{j=1}^{n}\left[c_j^l,c_j^u\right]x_j+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[q_{ij}^l,q_{ij}^u\right]x_ix_j$$
$$s,t,\begin{cases}\sum_{j=1}^{n}\left[a_{ij}^l,a_{ij}^u\right]x_j\geq\left[b_i^l,b_i^u\right],\ i=1,\dots,m\\ x_j\geq 0,\ j=1,\dots,n\end{cases} \tag{8}$$

But, in this paper the solution of the following IQP is dealt with:

$$\text{Min } Z=\sum_{j=1}^{n}c_jx_j+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}q_{ij}x_ix_j$$
$$s,t,\begin{cases}\sum_{j=1}^{n}\left[a_{ij}^l,a_{ij}^u\right]x_j\geq\left[b_i^l,b_i^u\right],\ i=1,\dots,m\\ x_j\geq 0,\ j=1,\dots,n\end{cases} \tag{9}$$

in which the constraints coefficient and the elements of right-hand side vector, are all interval numbers.

The solution of interval quadratic programming

According to the operations of the interval number, each inequality in (9) can be transformed into 2^{n+1} inequality as follows:

$a_1x_1+a_2x_2+\cdots+a_nx_n\geq b,\ a_j\in[a_j^l,a_j^u],\ b\in[b^l,b^u].$

Let D_i stand for the set of solutions to the i-th inequality and:

$$\bar{D}=\bigcup_{i=1}^{2^{n+1}}D_i,\quad \underline{D}=\bigcap_{i=1}^{2^{n+1}}D_i$$

Definition 5: Suppose $\sum_{j=1}^{n}[a_j^l,a_j^u]x_j\geq[b^l,b^u]$

The inequality $\sum_{j=1}^{n}a_jx_j\geq b$ is called the characteristic formula of $\sum_{j=1}^{n}[a_j^l,a_j^u]x_j\geq[b^l,b^u]$, where $a_j\in\left[a_j^l,a_j^u\right]$ and $b\in[b^l,b^u]$.

Definition 6: For each constraint inequality $\sum_{j=1}^{n}[a_j^l,a_j^u]x_j\geq[b^l,b^u]$, if there exists one characteristic formula $\sum_{j=1}^{n}a_jx_j\geq b$, such that its set of solution is the same as $\bar{D}$ or $\underline{D}$, then this characteristic formula is called as maximum value range inequality or minimum value range inequality.

By the theory of HyPlane, it is easy to obtain the following theorem.

Theorem1: Suppose $\sum_{j=1}^{n}[a_j^l,a_j^u]x_j\geq[b^l,b^u]$.

Then $\sum_{j=1}^{n}a_j^ux_j\geq b^l$ and $\sum_{j=1}^{n}a_j^lx_j\geq b^u$ are maximum value range inequality and minimum range inequality for this constraint condition, respectively.

Proof: For proving that the term $\sum_{j=1}^{n} a_j^u x_j \geq b^l$ is the maximum value range of inequality $\sum_{j=1}^{n} [a_j^l, a_j^u] x_j \geq [b^l, b^u]$, it must be shown that D¯ is the solution set of characteristic formula.

Consider the following characteristic formula for $\sum_{j=1}^{n} [a_j^l, a_j^u] x_j \geq [b^l, b^u]$:

$$\sum_{j=1}^{n} a_j x_j \geq b, \tag{10}$$

where $a_j \in [a_j^l, a_j^u]$ and $b \in [b^l, b^u]$.

Now it can be written as:

$$a_j \leq a_j^u,\ \forall j \xrightarrow{x_j \geq 0} a_j x_j \leq a_j^u x_j,\ \forall j \rightarrow \sum_{j=1}^{n} a_j x_j \leq \sum_{j=1}^{n} a_j^u x_j \tag{11}$$

Considering (10) and (11) and also the fact that $b^l \leq b$ yields:

$$b^l \leq b \leq \sum_{j=1}^{n} a_j x_j \leq \sum_{j=1}^{n} a_j^u x_j$$

Consequently, each solution of characteristic formula $\sum_{j=1}^{n} a_j x_j \geq b$ satisfies $\sum_{j=1}^{n} a_j^u x_j \geq b^l$.

Therefore, $\sum_{j=1}^{n} a_j^u x_j \geq b^l$ is the maximum value range.

Similarly, for proving that the term $\sum_{j=1}^{n} a_j^l x_j \geq b^u$ is the minimum value range of inequality $\sum_{j=1}^{n} [a_j^l, a_j^u] x_j \geq [b^l, b^u]$, it must be shown that $\underline{D}$ is the solution set of characteristic formula.

$$a_j^l \leq a_j,\ \forall j \xrightarrow{x_j \geq 0} a_j^l x_j \leq a_j x_j,\ \forall j \rightarrow \sum_{j=1}^{n} a_j^l x_j \leq \sum_{j=1}^{n} a_j x_j \tag{12}$$

Since $\sum_{j=1}^{n} a_j^l x_j \geq b^u$, the relation (12) and $b \leq b^l$ yield:

$$b \leq b^l \leq \sum_{j=1}^{n} a_j^l x_j \leq \sum_{j=1}^{n} a_j x_j$$

Consequently, each solution of $\sum_{j=1}^{n} a_j^l x_j \geq b^u$ satisfies characteristic formula $\sum_{j=1}^{n} a_j x_j \geq b$. Thus $\sum_{j=1}^{n} a_j^l x_j \geq b^u$ is the minimum value range. So the proof is completed.

According to this theorem for each constraint condition in interval number of QP (9), there is maximum value range inequality $\sum_{j=1}^{n} a_{ij}^u x_j \geq b_i^l$, and minimum value range inequality $\sum_{j=1}^{n} a_{ij}^l x_j \geq b_i^u$.

Then the interval number of QP (9) is reduced to the two following classical QP problems:

$$\begin{aligned} &\text{Min } Z' = \sum_{j=1}^{n} c_j x_j + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} x_i x_j \\ &s,t, \begin{cases} \sum_{j=1}^{n} a_{ij}^u x_j \geq b_i^l, & i = 1, \dots, m \\ x_j \geq 0, & j = 1, \dots, n \end{cases} \end{aligned} \tag{13}$$

$$\begin{aligned} &\text{Min } Z'' = \sum_{j=1}^{n} c_j x_j + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} x_i x_j \\ &s,t, \begin{cases} \sum_{j=1}^{n} a_{ij}^l x_j \geq b_i^u, & i = 1, \dots, m \\ x_j \geq 0, & j = 1, \dots, n \end{cases} \end{aligned} \tag{14}$$

Suppose the optimal solutions to (13) and (14) are $x' = (x_1', x_2', \dots, x_n')$ and $x'' = (x_1'', x_2'', \dots, x_n'')$ with the values of Z' and Z'', respectively. It is easy to prove the optimal value of (9) belongs to the interval $[Z', Z'']$, but the prove is omitted here.

However, since all numbers involved in model (13) and (14), are real numbers, these models are classical QP problems which can be solved using SQP algorithm in MATLABTM toolbox. The SQP algorithm is used as an optimization method to minimize the nonlinear constrained optimization problem. This method is described in the next subsection.

The SQP algorithm

SQP is an iterative analytical nonlinear programming method. This technique begins from an initial point to find a solution using the gradient based information. This optimization method is found faster than other population based search algorithms. Although the SQP method is highly dependent on the initial estimate of the solution (Bayo, Grau, Ruiz, & Sua´rez, 2010), this has successfully been applied in some optimal control problems. The SQP method is based on an iterative formulation together with the solution of some other QP sub-problems. An optimization problem in the SQP method is considered as follows:

$$\begin{aligned} &\text{Min } J(x) \\ &s,t,\ \psi_i(x) \leq 0,\ i = 1, \dots, m \end{aligned} \tag{15}$$

where $J(x)$ is the cost function and $\psi_i(x)$ stands for the constraint. In this regard, a Lagrangian function $L(x,\lambda)$ is constructed in terms of the Lagrangian multiplier λ_i. The cost function together with the above constraint is defined as follows:

$$L(x,\lambda) = J(x) + \sum_{i=1}^{m} \lambda_i \psi_i(x) \quad (16)$$

In fact, the SQP consists of three main parts:

1-Update the Hessian of the Lagrangian function according to:

$$H_{k+1} = H_k + \frac{q_k q_k^T}{q_k^T S_k} - \frac{H_k^T H_k}{S_k^T H_k S_k}$$

$$H_0 = I$$

$$S_k = X_{k+1} - X_k \quad (17)$$

$$q_k = \nabla f(X_{k+1}) + \sum_{i=1}^{n} \lambda_i \nabla g_i(X_{k+1}) - \left(\nabla f(X_k) + \sum_{i=1}^{n} \lambda_i \nabla g_i(X_k)\right)$$

2-Solve the QP sub-problem:

$$\min \frac{1}{2} q_k^T H_k d_k + \nabla f(x_k)^T d_k$$

$$\nabla \psi_i(x_k)^T d_k + \psi_i(x_k) = 0,\ i = 1, .. \quad (18)$$

$$\nabla \psi_i(x_k)^T d_k + \psi_i(x_k) \geq 0,\ i = 1,$$

3-A linear search to find a solution for the next iteration:

$$X_{k+1} = X_k + \alpha d_k \quad (19)$$

The algorithm is repeated until a stopping criterion (either maximum number of iterations or convergence criterion) is met. It must be mentioned that the SQP algorithm is a gradient based algorithm. Generally gradient based methods have the possibility of getting trapped at local optimum depending on the initial guess of the solution. In order to achieve a good final result, these methods require very good initial guesses of the solution. Since the matrix Q is supposed to be symmetric ($q_{ij}=q_{ji}$) and positive semi-definite, the objective function is convex and thus the SQP algorithm yields the global optimum solution. The corresponding theorem is presented as follows (Bazaraa, Sherali, & Shetty, 1993):

Theorem 2: In mathematical terminology, $f(x_1,x_2,\ldots,x_n)$ is convex if and only if its $n\times n$ Hessian matrix is positive semi-definite for all possible values of $(x_1,x_2,\ldots,x_n)$. That is for any $x\geq 0$ the following relation is satisfied:

$$(x_1, x_2, \ldots, x_n)\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_n x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \geq 0, \quad (20)$$

$$\forall (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

in which $H(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} = \left[\frac{\partial^2 f}{\partial x_j \partial x_i}\right]_{n\times n}$,

is the Hessian matrix of function f. Hence the relation (20) can be rewritten as follows:

$$x^T H(x) x \geq 0,\ \forall x \in \mathbb{R}^n \quad (21)$$

AN EXAMPLE

In this section, an example is presented to verify the performance of the proposed method. Consider a river from which diversions are made to three water–consuming firms that belong to the same corporation, as illustrated in Figure 1. Each firm makes a product, and is the critical resource. Water is needed in the process of making that product, and it is critical resource. The three firms can be denoted by the index $j=1,2,3$ and their water allocations by x_j. Assume the problem is to determine the allocations x_j of water to each of three firms ($j=1,2,3$) that maximize the total net benefits, $\sum_j NB_j(x_j)$, obtained from all three firms. The total amount of water available is constrained or limited to a quantity of Q. Assume the net benefits $NB_j(x_j)$, derived from water x_j allocated to each firm j, are defined by:

$$NB_1(x_1) = 3x_1 - x_1^2 \quad (22)$$

$$NB_2(x_2) = x_2 - x_2^2 \quad (23)$$

$$NB_3(x_3) = x_3 - x_3^2 \quad (24)$$

The problem is to find the allocations of water to each firm that maximize the total benefits $TB(X)$:

$$TB(X) = (3x_1 - x_1^2) + (x_2 - x_2^2) + (x_3 - x_3^2) \quad (25)$$

These allocations cannot exceed the amount of water available, Q, less any that must remain in the river, R. Assuming the available flow for allocations, Q-R, is 3. The crisp optimization problem is to maximize Equation (25) subject to the resource constraint:

$$x_1+x_2+x_3\leq 3 \tag{26}$$

Thus the problem is:

$$MaxTB(X)=(3x_1-x_1^2)+(x_2-x_2^2)+(x_3-x_3^2)$$

$$s.t.\ \begin{cases} x_1+x_2+x_3\leq 3 \\ x_1,\ x_2,x_3\geq 0 \end{cases} \tag{27}$$

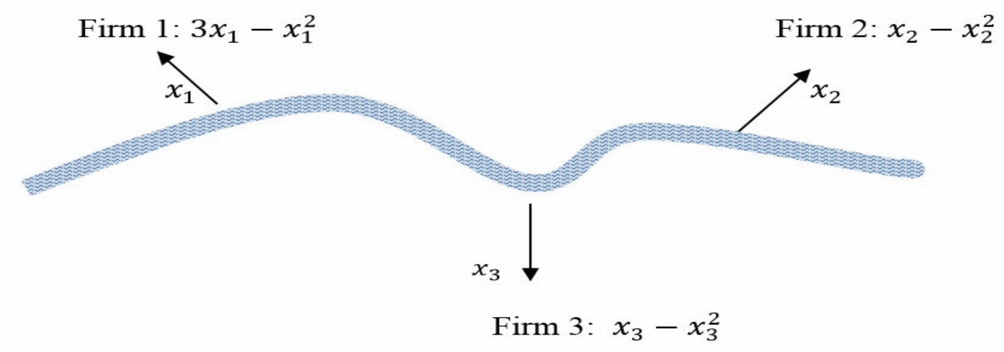

Fig.1

Now, assume that the available flow for allocations, Q-R, is not certainly known and is represented by an interval [1,5, 4,5]. Thus the problem turns to the IQP problem as follows:

$$Max\ TB(X)=(3x_1-x_1^2)+(x_2-x_2^2)+(x_3-x_3^2)$$

$$s.t.\ \begin{cases} x_1+x_2+x_3\leq [1.5,4.5] \\ x_1,\ x_2,x_3\geq 0 \end{cases} \tag{28}$$

That is equivalent to the following problem:

$$-Min\ \ TB(X)=-(3x_1-x_1^2)-(x_2-x_2^2)-(x_3-x_3^2)$$

$$s.t.\ \begin{cases} -x_1-x_2-x_3\geq [-4,5,-1,5] \\ x_1,\ x_2,x_3\geq 0 \end{cases} \tag{29}$$

Clearly this problem is in the form of model (9). Hence it can be solved using the proposed method. Since the parameter b_1, is interval number, the objective value of the problem should be interval number as well. According to Theorem 1, the minimum value range and maximum value range of interval solution obtained by solving two following programs respectively:

$$Z'=-Min\ \ TB(X)=-(3x_1-x_1^2)-(x_2-x_2^2)-(x_3-x_3^2)$$

$$s.t.\ \begin{cases} -x_1-x_2-x_3\geq -1.5 \\ x_1,\ x_2,x_3\geq 0 \end{cases}$$

$$Z''=-Min\ \ TB(X)=-(3x_1-x_1^2)-(x_2-x_2^2)-(x_3-x_3^2)$$

$$s.t.\ \begin{cases} -x_1-x_2-x_3\geq -4.5 \\ x_1,\ x_2,x_3\geq 0 \end{cases}$$

Where parameter values are all known constant. Thus, these models are conventional QP problems. By solving these problems using SQP algorithm the global optimum solutions are obtained as:

$$x'=(1,17,0,17,0,17),\ \ x''=(1,5,0,5,0,5)$$

The values of the minimum range and maximum range of the interval that objective value belongs to, are also achieved $Z'=2.42$ and $Z''=2.75$ respectively. Thus optimum value of IQP model (22) belongs to the interval [2.42,2,75].

CONCLUSION

Interval programming problems in comparison with fuzzy programming problems and probabilistic programming problems need to much less information. Also none of the interior numbers of interval has qualitative and insufficiency preference to other ones. Hence, it is so applicable and effective to use from interval programming in such situations. In spite of the fact that interval linear programming problem was noticed by many researchers as an interesting subject, no much observable development is happened in interval quadratic programming problem. Hence in this paper we deal with interval quadratic programming problem. This paper generalizes the conventional quadratic programming of constant parameters to interval parameters and the optimal interval value of the objective produced from the interval parameters, including constraint coefficients and right-hand sides. The idea is to reduce the interval quadratic programming problem to the two classical quadratic programming problems that yields the maximum range value and minimum range value of optimal interval value respectively. The numerical examination has been showed that the proposed method is so practical. We also emphasize that based on the proposed idea for solving this typical model, we may focus on sensitivity analysis on the parameters of the corresponding models. We are trying to extend this method to situations where parameters are trapezoidal or LR fuzzy numbers. Also we hope to extend our presented method for problems with fuzzy parameters and fuzzy variables to present better and comparative researches.

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