Theory of Approximation and Applications

Vol. 12, No.2, (2018), 115-129

Spectral Scheme for Solving Fuzzy Volterra Integral Equations of First Kind

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Received 15 July 2018; accepted 7 October 2018

Abstract

This paper discusses about the solution of fuzzy Volterra integral equation of the first-kind (F-VIE1) with nonlinear fuzzy kernels using spectral method. In this method the Gauss-Legendre points and Legendre weights are used to solve fuzzy Volterra integral equation of the first-kind (F-VIE1) with nonlinear fuzzy kernels. Indeed the parametric form of fuzzy driving term is applied for F-VIE1. Three classifications for F-VIE1 are searched to solve them, that these classifications are considered based on the interval sign of the kernel. Finally, two examples in two different cases are got to illustrate more. However, accuracy and efficiency are shown in tables.

Key words: Spectral method, Fuzzy Volterra integral equation of First-kind (F-VIE1), Fuzzy integral equation, Gauss-Legendre points.

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1 Introduction

The integral equations are seen in many problems in applied mathematic. The solutions of Integral equations are important for applying science, such as mechanics or physics. Many Integral equations have fuzzy parameters and a few methods expanded to fuzzy integral equations. The numerical methods are used widely in recent years that these can to obtained the solutions of with great accuracy.

Spectral scheme is a powerful method that uses basis functions which are nonzero over the domain such as Legendre polynomial. Actually spectral methods take on a global approach. Specially for this reason, spectral methods have excellent error properties, with the so-called exponential convergence being the fastest possible, when the solution is smooth, [7,**?**].

The concepts of fuzzy numbers were first introduced by Zadeh, [25] but Friedman and et.al introduced the numerical solution of the fuzzy integral equation by embedding method. There are some research over existence and unique of integral equations and sufficient conditions for convergence of their proposed method, [13,16,18,19,22,24]. Special Park and Jeong search exist and uniqueness of Fredholm-Volterra integral equation (F-VIE), [19]. Abbasbandy and et. al, used a parametric fuzzy number to convert a linear fuzzy Fredholm integral equation of second kind, such as the Nystrom approximation, [2]. Molabahrami et.al, by using the parametric form of the fuzzy number converted a linear fuzzy F-VIE to two linear systems of the second kind of integral equation in crisp case. They used the Homotopy analysis method to find the approximate solution of these systems, [17]. Ghanbari et.al, used the Block-pulse functions to approximate the numerical solution of fuzzy F-IE, [14]. Sadeghi goghari et. al, present two methods which exploit hybrid Legendre and Block Pulse functions and Legendre wavelets to find the approximate solution for a system of linear fuzzy F-IE of the second kind with two variables, [21]. Babolian et.al converted a linear fuzzy F-VIE of the second kind to a linear system of integral equation in the crisp case by Adomian method, [6]. Abbasbandy snd et. al applied spectral method for for Volterra-Fredholm integral equations of the first kind and proved the main theorem of spectral relationships and its application, [1].

In this paper the numerical method for solving the F-VIE1 equations in the form

$$\int_a^x K(x,s)\tilde{u}(s)ds = \tilde{g}(x), \qquad x \in [a,b] \tag{1.1}$$

is discussed. This numerical method is restricted to spectral method.

In section 2, some needed concepts are reviewed briefly . In section 3, we proposed our method to find solution by spectral method. In section 4, F-VIE1 is classified. Finally we get two example to illustrate more our method in section 5. In whole of paper $\tilde{a}$ is a notation for the fuzzy set of a.

2 Basic concepts

The basic definitions of a fuzzy number are given in [11,15,25,26] as follows:

Definition 2.1 *A fuzzy number is a fuzzy set like* $u : \mathbb{R} \to [0,1]$ *which satisfies:*

1. *u is an upper semi-continuous function,*
2. $u(x) = 0$ *outside some interval [a,d],*
3. *There are real numbers* b, c *such as* $a \leq b \leq c \leq d$ *and*
   3.1 $u(x)$ *is a monotonic increasing function on [a, b],*
   3.2 $u(x)$ *is a monotonic decreasing function on [c, d],*
   3.3 $u(x) = 1$ *for all* $x \in [b,c]$.

Definition 2.2 *A fuzzy number u in parametric form is a pair* $(\underline{u}, \overline{u})$ *of functions* $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, *which satisfy the following requirements:*

1. $\underline{u}(r)$ *is a bounded non-decreasing left continuous function in* $(0,1]$, *and right continuous at 0,*
2. $\overline{u}(r)$ *is a bounded non-increasing left continuous function in* $(0,1]$, *and right continuous at 0,*
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

Definition 2.3 *For arbitrary* $\tilde{u} = (\underline{u}(r), \overline{u}(r))$ *and* $\tilde{v} = (\underline{v}(r), \overline{v}(r))$, $0 \leq r \leq 1$, *and scalar k, we define addition, subtraction, scalar product by k and multiplication are respectively as following:*

$$\begin{array}{rll} addition: & \underline{u+v}(r)=\underline{u}(r)+\underline{v}(r), & \overline{u+v}(r)=\overline{u}(r)+\overline{v}(r) \\ subtraction: & \underline{u-v}(r)=\underline{u}(r)-\overline{v}(r), & \overline{u-v}(r)=\overline{u}(r)-\underline{v}(r) \end{array}$$

$$\text{scalar product: } k\tilde{u}=\begin{cases}(k\underline{u}(r),k\overline{u}(r)), & k\geq 0\\ (k\overline{u}(r),k\underline{u}(r)), & k<0\end{cases} \quad multiplication: \quad \underline{uv}(r)=$$
$$max\{\underline{u}(r)\underline{v}(r),\underline{u}(r)\overline{v}(r),\overline{u}(r)\underline{v}(r)\overline{u}(r)\overline{v}(r)\}$$
$$\overline{uv}(r)=min\{\underline{u}(r)\underline{v}(r),\underline{u}(r)\overline{v}(r),\overline{u}(r)\underline{v}(r)\overline{u}(r)\overline{v}(r)\}$$

Definition 2.4 *The metric structure is given by Hausdorff distance*
$D:\mathbb{R}_F\times\mathbb{R}_F\longrightarrow\mathbb{R}_+\cup 0$
$D(u(r),v(r))=Max\{sup|\underline{u}-\underline{v}|,sup|\overline{u}-\overline{v}|\}$
$(\mathbb{R}_F,D)$ *is a complete metric space and following properties are well known:*
$D(u+w,v+w)=D(u,v),\quad \forall u,v,w\in\mathbb{R}_F$
$D(ku,kv)=|k|D(u,v),\quad \forall u,v\in\mathbb{R}_F,\quad \forall k\in\mathbb{R}$
$D(u+v,w+e)\leq D(u,w)+D(v,e),\quad \forall u,v,w,e\in\mathbb{R}_F$

If the fuzzy function $f(x)$ is continuous in the metric D, its definite integral exists. Furthermore

$$\underline{(\int_a^b f(x)dx)}=(\int_a^b \underline{f}(x)dx)$$

$$\overline{(\int_a^b f(x)dx)}=(\int_a^b \overline{f}(x)dx)$$

For arithmetic in overall $s\in[a,b]$ for the following equation

$$\tilde{f}(x)=\int_a^x K(x,s)\tilde{u}(s)ds,$$

can be transform to two equations:

$$\underline{f}(x)=\int_a^x v_1(x,s,\underline{u}(s),\overline{u}(s))ds$$

$$\overline{f}(x)=\int_a^x v_2(x,s,\underline{u}(s),\overline{u}(s))ds$$

which

$$v_1(s,t,\underline{u}(s,r),\overline{u}(s,r)) = \begin{cases} k(x,s)\underline{u}(s), & k(x,s) \geq 0 \\ k(x,s)\overline{u}(s), & k(x,s) < 0 \end{cases}$$

and

$$v_2(s,t,\underline{u}(s,r),\overline{u}(s,r)) = \begin{cases} k(x,s)\overline{u}(s), & k(x,s) \geq 0 \\ k(x,s)\underline{u}(s), & k(x,s) < 0 \end{cases}$$

Definition 2.5 *For simplify in arithmetics over parametric form of fuzzy number we define: Let $u(r) = [\underline{u}(r), \overline{u}(r)], 0 \leq r \leq 1$ be a fuzzy number we take*

$$u^c = \frac{\underline{u}(r) + \overline{u}(r)}{2}$$

$$u^d = \frac{\overline{u}(r) - \underline{u}(r)}{2}$$

It is clear that $u^d(r) \geq 0$ and $\underline{u}(r) = u^c(r) - u^d(r)$ and $\overline{u}(r) = u^c(r) + u^d(r)$

Definition 2.6 *Let $u(r) = [\underline{u}(r), \overline{u}(r)], v(r) = [\underline{v}(r), \overline{v}(r)], 0 \leq r \leq 1$ are two fuzzy numbers and also k, s are two arbitrary real numbers. If $w = ku + sv$ then by using definition (2.5)*

$$w^c(r) = ku^c(r) + sv^c(r)$$

$$w^d(r) = |k|u^d(r) + |s|v^d(r)$$

3 Fuzzy Legendre-collocation method

Consider the first-kind Volterra fuzzy integral equation in the form (1). The collocation points are chosen as the set of $(N+1)$ Gauss-Legendre $\{x_i\}_{i=0}^N$. Then Eq. (1) is transformed to the following equation:

$$\int_{-1}^{x_i} K(x_i, s)\tilde{u}(s)ds = \tilde{g}(x_i), \qquad 0 \leq i \leq N \tag{3.1}$$

Now apply the following linear transformation:

$$s(x,\theta) = \frac{x+1}{2}\theta + \frac{x-1}{2}, \qquad -1 \le \theta \le 1 \tag{3.2}$$

then Eq. (3) becomes as follows:

$$\frac{1+x_i}{2}\int_{-1}^{1} K(x_i, s(x_i,\theta))\tilde{u}(s(x_i,\theta))d\theta = \tilde{g}(x_i), \qquad 0 \le i \le N \tag{3.3}$$

Using $(N+1)-$point Gauss-Legendre quadrature formula relative to the Legendre weights $\{\omega_k\}$ gives the bottom equation:

$$\frac{1+x_i}{2}\sum_{j=0}^{N} K(x_i, s(x_i,\theta))\tilde{u}(s(x_i,\theta))\omega_j = \tilde{g}(x_i), \qquad 0 \le i \le N \tag{3.4}$$

which $\tilde{u}$ is estimated by Lagrange interpolation polynomials as following sentences:

$$\tilde{u}(\sigma) \approx \sum_{j=0}^{N} \tilde{u}_j l_j(\sigma) \tag{3.5}$$

where the l_j is j-th Lagrange basic function. Combination Eqs. (6) and (7) yields:

$$\frac{1+x_i}{2}\sum_{j=0}^{N} \tilde{u}_j (\sum_{p=0}^{N} K(x_i, s(x_i,\theta))l_p(s(x_i,\theta))\omega_p) = \tilde{g}(x_i), \quad 0 \le i \le N \tag{3.6}$$

If $K(x,s)$ is continuous function over $[-1,1]$, then respect to the sign of $K(x,s)$, three cases can be considered that searched in the next section.

4 The classification of F-VIE1

By using sign of $K(x,s)$ over $[-1,1]$ three cases are got, that are perused them. The parametric form of equation (1) can be write in following:

$$\int_{-1}^{x} K(x,s)[\overline{u}(x), \underline{u}(s)]ds = [\overline{g}(x), \underline{g}(x)], \qquad x \in [-1,1] \tag{4.1}$$

$\underline{u}(r)$ and $\overline{u}(r)$ for all $0 \le r \le 1$ and for $x \in [-1,1]$ by Lagrange polynomials can be consider in following terms:

$$\underline{u}(x) \approx \sum_{j=0}^{N} \underline{u}_j(r) l_j(x)$$

$$\overline{u}(x) \approx \sum_{j=0}^{N} \overline{u}_j(r) l_j(x)$$

4.1 Case (1)

If $K(x,s)$ is positive over $[-1,1]$ Eq. (1) is transformed to

$$\int_{-1}^{x} K(x,s)\underline{u}(s)ds = \underline{g}(x), \qquad x \in [-1,1] \tag{4.2}$$

and

$$\int_{-1}^{x} K(x,s)\overline{u}(s)ds = \overline{g}(x), \qquad x \in [-1,1] \tag{4.3}$$

Now by using Eq. (4), consider Eqs. (9) and (10) in following respectively:

$$\frac{1+x_i}{2}\int_{-1}^{1} K(x_i, s(x_i,\theta))\underline{u}(s(x_i,\theta))(r)d\theta = \underline{g}(x_i)(r), \qquad 0 \le i \le N \tag{4.4}$$

$$\frac{1+x_i}{2}\int_{-1}^{1} K(x_i, s(x_i,\theta))\overline{u}(s(x_i,\theta))(r)d\theta = \overline{g}(x_i)(r), \qquad 0 \le i \le N \tag{4.5}$$

by using the set of $(N+1)$ Gauss-Legendre $\{x_i\}_{i=0}^{N}$, the Eqs. (11) and (12) transform to the following equations respectively:

$$\underline{u}_i(r) = \frac{1+x_i}{2}\sum_{j=0}^{N} \underline{u}_j(r)\Big(\sum_{p=0}^{N} K(x_i, s(x_i,\theta)) l_p(s(x_i,\theta))\omega_p\Big), \qquad 0 \le i \le N \tag{4.6}$$

$$\overline{u}_i(r) = \frac{1+x_i}{2}\sum_{j=0}^{N} \overline{u}_j(r)\Big(\sum_{p=0}^{N} K(x_i, s(x_i,\theta)) l_p(s(x_i,\theta))\omega_p\Big), \qquad 0 \le i \le N \tag{4.7}$$

Now two crisp equations of the matrix form (13) and (14) must to solve and $\underline{u}(r)$ and $\overline{u}(r)$ for all $0 \le r \le 1$ and $x \in [-1,1]$ are got.

4.2 Case (2)

In this case, hypothesis $K(x,s)$ is negative over $[-1,1]$. Then Eq. (5) is transformed to two following equations:

$$\underline{u}(x_i)(r) = \frac{1+x_i}{2}\int_{-1}^{1} K(x_i, s(x_i,\theta))\overline{u}(s(x_i,\theta))(r)d\theta, \qquad 0 \le i \le N \quad (4.8)$$

$$\overline{u}(x_i)(r) = \frac{1+x_i}{2}\int_{-1}^{1} K(x_i, s(x_i,\theta))\underline{u}(s(x_i,\theta))(r)d\theta, \qquad 0 \le i \le N \quad (4.9)$$

then put (4) in (15) and (16), we holds:

$$\underline{u}_i(r) = \frac{1+x_i}{2}\sum_{j=0}^{N}\overline{u}_j(r)(\sum_{p=0}^{N} K(x_i, s(x_i,\theta))l_p(s(x_i,\theta))\omega_p), \qquad 0 \le i \le N \quad (4.10)$$

$$\overline{u}_i(r) = \frac{1+x_i}{2}\sum_{j=0}^{N}\underline{u}_j(r)(\sum_{p=0}^{N} K(x_i, s(x_i,\theta))l_p(s(x_i,\theta))\omega_p), \qquad 0 \le i \le N \quad (4.11)$$

Now if we get

$$A_{i,j} = \frac{1+x_i}{2}\sum_{j=0}^{N} K(x_i, s(x_i,\theta))l_j(s(x_i,\theta))\omega_j, \qquad 0 \le i \le N \quad (4.12)$$

then to find solution in (17) and (18) and using definitions (2.5) and (2.6) for all $0 \le i \le N$ we have

$$u_i^c(r) = A_{ij}\sum_{p=0}^{N} u_p^c(r) \quad (4.13)$$

$$u_i^d(r) = A_{ij}\sum_{p=0}^{N} u_p^d(r) \quad (4.14)$$

Now two crisp equations of the matrix form (20) and (21) can to solve. Using definition (2.6) we can obtain $\tilde{u}_i = [\underline{u}_i, \overline{u}_i]$ for $i = 0, 1, ..., N$. When the values of $\tilde{u}_i = [\underline{u}_i, \overline{u}_i]$ for $i = 0, 1, ..., N$, are resulted the numerical solution for $x \in [-1,1]$ can be obtained by Lagrange polynomials.

4.3 Case (3)

In case (3), suppose that $K(x,s)$ is continuous in $-1 \leq s \leq 1$ and for fix x, the sign of $K(x,s)$ changes in finite points as t_i, for example without lose generality $K(x,s)$ is positive over $[-1,t]$ and negative over $[t,x]$, we have

$$\tilde{u} = \int_{-1}^{t} K(x,s)\tilde{u}(s)ds + \int_{t}^{x} K(x,s)\tilde{u}(s)ds \tag{4.15}$$

then by (N+1) Gauss-Legendre points we can write:

$$\tilde{u}(x_i) = \int_{-1}^{t} K(t,s)\tilde{u}(s)ds + \int_{t}^{x_i} K(x_i,s)\tilde{u}(s)ds \tag{4.16}$$

Now it can transform interval $[t,1]$ to $[-1,1]$ by

$$s = \frac{x_i + t}{1-t}\eta + \frac{x-1}{1-t}, -1 \leq \eta \leq 1 \tag{4.17}$$

by (4) and (24) we have

$$\tilde{u}(x_i) = \frac{1+t}{2}\int_{-1}^{1} K(t,s(t,\theta))\tilde{u}(s)d\theta + \frac{x_i+t}{1-t}\int_{-1}^{1} K(x_i,s(t,\eta))\tilde{u}(s)d\eta \tag{4.18}$$

Now by using Legendre weights, the Eq. (25) is transformed to

$$\tilde{u}_i = \frac{1+t}{2}\sum_{j=0}^{N}(\tilde{u}_j\sum_{j=0}^{N} K(t,s(t,\theta))l_j(s(t,\theta))\omega_j) + \frac{x_i+t}{1-t}\sum_{j=0}^{N}(\tilde{u}_j\sum_{j=0}^{N} K(x_i,s(t,\eta))l_j(s(x_i,\eta))\omega_j) \tag{4.19}$$

then with using the sign of K on the intervals we have

$$\underline{u}_i(r) = \frac{1+t}{2}\sum_{j=0}^{N}(\underline{u}_j(r)\sum_{j=0}^{N} K(t,s(t,\theta))l_j(s(t,\theta))\omega_j) + \frac{x_i+t}{1-t}\sum_{j=0}^{N}(\overline{u}_j(r)\sum_{j=0}^{N} K(x_i,s(t,\eta))l_j(s(x_i,\eta))\omega_j) \tag{4.20}$$

$$\overline{u}_i(r) = \frac{1+t}{2}\sum_{j=0}^{N}(\overline{u}_j(r)\sum_{j=0}^{N} K(t,s(t,\theta))l_j(s(t,\theta))\omega_j) + \frac{x_i+t}{1-t}\sum_{j=0}^{N}(\underline{u}_j(r)\sum_{j=0}^{N} K(x_i,s(t,\eta))l_j(s(x_i,\eta))\omega_j) \tag{4.21}$$

then we can write following matrices:

$$u_i^c(r) = (B_j + C_{ij}) \sum_{p=0}^{N} u_p^c(r), \qquad 0 \le i \le N \tag{4.22}$$

$$u_i^d(r) = (B_j - C_{ij}) \sum_{p=0}^{N} u_p^d(r), \qquad 0 \le i \le N \tag{4.23}$$

which

$$B_j = \frac{1+t}{2} \sum_{j=0}^{N} K(t, s(t,\theta)) l_j(s(t,\theta)) \omega_j$$

$$C_{ij} = \frac{x_i + t}{1-t} \sum_{j=0}^{N} K(x_i, s(t,\eta)) l_j(s(x_i,\eta)) \omega_j$$

By using definition (2.5) we can obtain $\tilde{u}_i = [\underline{u}_i, \overline{u}_i]$ for $i = 0, 1, ..., N$. When the values of $\tilde{u}_i = [\underline{u}_i, \overline{u}_i]$ for $i = 0, 1, ..., N$, are the numerical solution for $x \in [-1, 1]$ can be obtained by Lagrange polynomials.

Now in Eqs. (13), (14), (20), (21), (29) and (30), we use the spectral collocation algorithm to solve them. We get spectral collocation algorithm in the next section.

Now spectral collocation method is applied: Denoting $U_N = \{u_0, u_1, ..., u_N\}^t$. $L_j(s)$ is expressed in Legendre functions

$$l_j(s) = \sum_{p=0}^{N} \alpha_{p,j} L_p(s) \tag{4.24}$$

where $\alpha_{p,j}$ is discrete polynomial coefficients. The inverse relation

$$\alpha_{p,j}(s) = \frac{1}{\gamma_p} \sum_{i=0}^{N} l_j(x_i) L_p(x_i) \omega_i = \frac{L_p(x_j)\omega_j}{\gamma_p} \tag{4.25}$$

where

$$\gamma_p = \sum_{i=0}^{N} L_p^2(x_i)\omega_i = (p + \frac{1}{2})^{-1}, p < N \tag{4.26}$$

and $\gamma_N = (N + \frac{1}{2})^{-1}$ for the Gauss formulas.

$$l_j(s) = \sum_{p=0}^{N} \frac{L_p(x_j)L_p(s)\omega_j}{\gamma_p} \tag{4.27}$$

Consider $\underline{U}_N = \{\underline{u}_1, \underline{u}_2, ...\underline{u}_N\}$ and $\overline{U}_N = \{\overline{u}_1, \overline{u}_2, ...\overline{u}_N\}$ instead of U_N and $\underline{g}_N = \{\underline{g}(x_1), \underline{g}(x_2), ...\underline{g}(x_N)\}$ and $\overline{g}_N = \{\overline{g}(x_1), \overline{g}(x_2), ...\overline{g}(x_N)\}$ instead of g_N. Then in Eqs. (13), (14), (20), (21), (29) and (30), can be used the spectral collocation algorithm to solve them. The Legendre-Gauss-Lobatto points that are zeros of $L'(N+1)(x)$ are used as the collocation points. For the Legendre- Gauss-Lobatto points, the corresponding weights are

$$w_j = \frac{2}{(1-x_j^2)[L'_{N+1}(x_j)]^2}, 0 \leq j \leq N.$$

5 Numerical Example

In this section two examples are presented and solved by Legendre-spectral method and tables and figures are presented. The estimate and exact solutions are compared in examples and different between them are computed.

Example 5.1 *Consider the following fuzzy Volterra integral equation of first kind:*

$$[r\frac{x^5}{5}(1-2x), (2-r)\frac{x^5}{5}(1-2x)] = \int_0^x s^2(1-2x)u(s)ds$$

This equation is in case (3). The exact solution is $u(t) = [rt^2, (2-r)t^2]$. *Spectral scheme is applied for it. The absolute errors for* $\underline{u}(t)$ *and* $\overline{u}(t)$ *are shown in Table (5.1). These results indicate that the desired spectral accuracy is obtained for* $N = 12$ *and* $N = 20$ *with* $x = 0.1$ *and* $x = 2$.

Example 5.2 *Consider the following fuzzy Volterra integral equation:*

$$[(r-1)(x^2cosx - xsinx-), (1-r)(x^2cosx - xsinx)] = \int_0^x -x^2u(s)ds$$

r	$\lvert Err(\underline{u})\rvert, N = 12, x = 0.1$	$\lvert Err(\underline{u})\rvert, N = 20, x = 0.1$	$\lvert Err(\underline{u})\rvert, N = 12, x = 2$	$\lvert Err(\underline{u})\rvert, N = 20, x = 2$	$\lvert Err(\overline{u})\rvert, N = 12, x = 0.1$	$\lvert Err(\overline{u})\rvert, N = 20, x = 0.1$	$\lvert Err(\overline{u})\rvert, N = 12, x = 2$	$\lvert Err(\overline{u})\rvert, N = 20, x = 2$
0	0	0	0	5.233E(-11)	0.00001213	3.199E(-7)	3.199E(-7)	1.348E(-11)
0.1	0.000065327	3.945E(-7)	4.302E(-7)	3.323E(-11)	0.00003732	2.158E(-7)	3.199E(-7)	3.232E(-11)
0.2	0.000031271	2.032E(-7)	2.981E(-7)	4.013E(-11)	0.00002543	4.332E(-7)	3.199E(-7)	2.299E(-11)
0.3	0.000042754	2.103E(-7)	4.433E(-7)	3.857E(-11)	0.00001263	3.123E(-7)	3.199E(-7)	1.442E(-11)
0.4	0.000034391	3.325E(-7)	3.176E(-7)	3.409E(-11)	0.00002643	1.329E(-7)	3.199E(-7)	2.324E(-11)
0.5	0.000038423	2.333E(-8)	2.231E(-7)	3.109E(-12)	0.00005342	3.231E(-7)	3.199E(-7)	2.124E(-11)
0.6	0.000022329	3.311E(-7)	3.045E(-7)	2.347E(-12)	0.00003254	4.262E(-7)	3.199E(-7)	1.324E(-11)
0.7	0.000035432	2.223E(-7)	5.423E(-7)	2.653E(-12)	0.00002365	3.324E(-7)	3.199E(-7)	2.324E(-11)
0.8	0.000042186	3.235E(-7)	2.423E(-7)	2.143E(-12)	0.00001437	2.173E(-7)	3.199E(-7)	4.324E(-11)
0.9	0.000032987	3.204E(-7)	1.134E(-7)	1.834E(-12)	0.00002354	3.373E(-7)	3.199E(-7)	5.621E(-11)
1	0.000027654	2.991E(-7)	1.681E(-8)	1.634E(-12)	0.00003254	1.275E(-7)	3.199E(-7)	3.523E(-11)

Table 5.1: Error results for $x = 0.1$ and $x = 2$ with $N = 12$ and $N = 20$ for $\underline{u}$ and $\overline{u}$ in Example (5.1)

r	$\lvert Err(\underline{u})\rvert, N = 12, x = 0.2$	$\lvert Err(\underline{u})\rvert, N = 20, x = 0.2$	$\lvert Err(\underline{u})\rvert, N = 12, x = 1$	$\lvert Err(\underline{u})\rvert, N = 20, x = 1$	$\lvert Err(\overline{u})\rvert, N = 12, x = 0.2$	$\lvert Err(\overline{u})\rvert, N = 20, x = 0.2$	$\lvert Err(\overline{u})\rvert, N = 12, x = 1$	$\lvert Err(\overline{u})\rvert, N = 20, x = 1$
0	2.245E(-3)	2.375E(-6)	0	5.3E(-13)	3.232E(-3)	1.320E(-6)	4.001E(-8)	3.438E(-14)
0.1	3.278E(-3)	1.163E(-6)	4.334E(-7)	4.812E(-13)	1.239E(-3)	5.345E(-6)	4.102E(-8)	2.148E(-14)
0.2	2.256E(-2)	1.263E(-6)	3.198E(-7)	4.543E(-14)	2.335E(-3)	5.123E(-6)	4.033E(-8)	2.245E(-14)
0.3	5.225E(-3)	1.327E(-6)	4.543E(-7)	3.876E(-14)	1.543E(-3)	5.373E(-6)	3.345E(-8)	3.128E(-14)
0.4	2.187E(-2)	1.735E(-6)	1.213E(-7)	3.314E(-14)	4.232E(-3)	5.230E(-6)	1.149E(-8)	4.018E(-14)
0.5	5.983E(-2)	1.318E(-5)	2.543E(-8)	3.1181E(-13)	3.523E(-3)	3.243E(-6)	3.001E(-7)	3.132E(-14)
0.6	2.345E(-2)	1.316E(-5)	3.543E(-7)	2.243E(-14)	2.221E(-3)	6.303E(-6)	4.202E(-7)	4.206E(-14)
0.7	1.105E(-3)	1.443E(-5)	5.129E(-7)	2.343E(-14)	3.271E(-3)	2.334E(-5)	3.465E(-7)	3.398E(-14)
0.8	2.551E(-3)	1.322E(-5)	8.324E(-7)	2.129E(-14)	3.193E(-3)	5.113E(-6)	2.334E(-7)	3.343E(-14)
0.9	2.007E(-2)	1.543E(-5)	2.154E(-7)	1.033E(-14)	2.443E(-3)	1.033E(-6)	3.133E(-7)	2.178E(-14)
1	3.142E(-3)	1.319E(-5)	1.643E(-8)	1.022E(-13)	1.287E(-3)	2.233E(-6)	1.151E(-7)	1.067E(-14)

Table 5.2: Error results for $x = 0.2$ and $x = 1$ with $N = 12$ and $N = 20$ for $\underline{u}$ and $\overline{u}$ in Example (5.2)

This equation is in case (2). The exact solution is $u(x) = [(r-1)sins, (1-r)sins]$. The absolute errors for $\underline{u}(t)$ and $\overline{u}(t)$ are shown in Table (5.2). These results indicate that the desired spectral accuracy is obtained for $N = 12$ and $N = 20$ with $x = 0.2$ and $x = 1$.

6 Conclusion

In this work, fuzzy Volterra integral equations of first kind were studied. The spectral scheme was successfully employed for solving it, based on Legendre points and Lagrange interpolation polynomials. The obtained results was illustrated very near to the exact solutions. It was shown that this technique is useful to implement to producing accurate results and the accuracy of obtained data of spectral method is high. This method can be extended to fuzzy Volterra integro-differential equations with linear fuzzy kernels with some needed modifications. The computations in this paper were performed by using Maple 18.

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