

# Two-Dimensional Elasticity Solution for Arbitrarily Supported Axially Functionally Graded Beams

A. Singh, P. Kumari \*

*Department of Mechanical Engineering, Indian Institute of Technology Guwahati, Guwahati 781039, India*

Received 5 July 2018; accepted 8 September 2018

## ABSTRACT

First time, an analytical two-dimensional (2D) elasticity solution for arbitrarily supported axially functionally graded (FG) beam is developed. Linear gradation of the material property along the axis of the beam is considered. Using the strain displacement and constitutive relations, governing partial differential equations (PDEs) is obtained by employing Rissiner mixed variational principle. Then PDEs are reduced to two set of ordinary differential equations (ODEs) by using recently developed extended Kantorovich method. The set of  $4n$  ODEs along the  $z$ -direction has constant coefficients. But, the set of  $4n$  nonhomogeneous ODEs along  $x$ -direction has variable coefficients which is solved using modified power series method. Efficacy and accuracy of the present methodology are verified thoroughly with existing literature and 2D finite element solution. Effect of axial gradation, boundary conditions and configuration lay-ups are investigated. It is found that axial gradation influence vary with boundary conditions. These benchmark results can be used for assessing 1D beam theories and further present formulation can be extended to develop solutions for 2D micro or Nanobeams. © 2018 IAU, Arak Branch. All rights reserved.

**Keywords :** Axially functionally graded; Two-Dimensional elasticity; Arbitrary supported; Extended Kantorovich method.

## 1 INTRODUCTION

VARIOUS engineering structures such as turbine blades, ship propellers, robot arms, helicopter rotor blades and space structures are often modeled as a beam. Functionally graded (FG) beams offer high thermal resistance, high toughness, low density and low-stress concentration as compared to composite laminated beams. In FG beams, material properties vary gradually along the spatial coordinates (either axial or through the thickness). This gradation in material properties may due to some environment effects like corrosion, high temperature, exposed to the gaseous environment [1–3], or it may induce intentionally during fabrication to meet design requirements [4]. Therefore, design and development of axially graded beams have become topic of research from last decade. There are number of analytical, semi-analytical and numerical techniques have been developed to predicts the behavior of through-thickness functionally graded beams [5–16] based on elasticity theory or one-dimensional beam theories like Euler-Bernoulli beam theory, Timoshenko beam theory and Reddy or higher order shear deformation theories. However, there are certain circumstances such as graded temperature field, non-uniform loading where the axially graded beam can resist multi-directional sever variation of loading better than through thickness graded beams. For

\*Corresponding author. Tel.: +91 361 2583434.  
E-mail address: [kpmech@iitg.ernet.in](mailto:kpmech@iitg.ernet.in) (P.Kumari).

an axially graded beam, exact close form solution is difficult to obtain due to the presence of variable coefficients in the governing equations. Therefore, very few research articles are presented in literature. First-time, Elishakoff and Candan [17] introduced a closed-form solution of axially grade beam using Euler-Bernoulli beam theory. The inverse method of solution was applied to obtain the solution for free vibration case. Huang and Li [18] developed a dynamic solution for axially graded non-uniform Euler-Bernoulli beam by employing Fredholm integral equations. Giunta et al. [19] presented a linear static analysis of simply supported beams having axially or bi-directional material gradation based on refined theories. Sarkar and Ganguli [20] developed closed-form dynamic solution for non-uniform axially graded Euler-Bernoulli beam and further extended this approach to analyze axially graded Timoshenko beams [21]. Li et al. [22] presented frequency analysis of Euler-Bernoulli beam in which density and stiffness vary exponentially along the axis of beam. Tang and Wu [23] presented exact solutions for arbitrary supported Timoshenko beam which has exponentially gradation along the longitudinal axis of beam. Nguyen et al. [24] presented an analytical solution for functionally graded beams (axially as well as through-thickness) with tapered cross-section under static loading by taking power-law variation in elastic modulus of the beam. Recently, Kukla and Rychlewska [25] developed a free vibration solution for axially graded beam having non-uniform cross-section and arbitrary axial inhomogeneity. Very recently, Huang and Rong [26] analyzed the dynamic behavior of nonuniform axially graded Euler-Bernoulli beam using power series method. In this work, material property follows random gradation along the axial direction of the beam.

The numerical methods are also employed to develop solutions for axially FG beams. Shahba et al. [27, 28] and Shahba and Rajasekaran [29] analyzed dynamic response and stability of Timoshenko and Euler-Bernoulli axially graded beams by employing finite element methods and other numerical methods, e.g., differential transforms element method (DTEM). Li et al. [30] presented solution for bending and free vibration of axially and transversally functionally graded (FG) beam with variable cross-section by employing new finite element technique, in which exact displacement interpolation function was used. Arefi and Rahimi [31] presented an analytical solution of the functionally graded beam having variable thickness based on Euler Bernoulli theory. Giunta et al. [32] presented three-dimensional static bending analysis of functionally graded beams by employing hierarchical modeling and a collocation meshless method. Recently, Arefi and Zenkour [33] developed solution for free vibration and bending of a sandwich microbeam based on first-order shear deformation theory and further extended the approach for the analysis of the piezo-magnetic three layers nano-beams based on a combination of nonlocal model and Timoshenko model [34]. Further, Zenkour et al. [35] developed solution for sandwich curved nano-beam integrated with piezo-magnetic face-sheets using non-local theory to investigate the effect of size. Using the non-local theory, Arefi and Zenkour [36] presented the bending analysis of sandwich nano-beams having functionally graded core and piezo-magnetic face sheets. Using GDQM (Generalized differential quadrature method), Li et al. [37] analyzed the bending, buckling and vibration behaviors of size-dependent axially functionally graded beams based on Euler-Bernoulli beam theory. The material properties follow power-law variation along the axis of the beam. Most of the work for the axially graded or homogeneous beam are based on 1D theories. Two-dimensional elasticity solution is always needed for predicting accurate behavior and boundary effects of composite laminated or functionally graded beams [38]. Further, the closed-form 2D elasticity results help to verify the accuracy and efficacy of 1D analytical, approximate and numerical solutions. Recently, Kapuria and coworkers [39, 40] presented analytical solution for arbitrary supported plate based on three-dimensional (3D) elasticity by employing the extended Kantorovich method. Very recently, Kumari et al. [41] extended this approach to develop a 3D analytical solution for static analysis of Levy-type functionally graded plate having in-plane stiffness variation. As per author knowledge, there is no closed-form 2D elasticity solution for axially functionally graded beams exist in the literature. Moreover, 2D elasticity the analytical solution for the arbitrary supported homogeneous beam is also not available in the literature. Hence, the present work will help to fill up the gap in the literature by providing analytical two-dimensional elasticity solution for arbitrary supported axially functionally graded beam. Further, two-dimensional elasticity solution for the arbitrary supported homogeneous beam is obtained as a special case of present study.

A benchmark two-dimensional (2D) elasticity analytical solution is presented for axially functionally graded beam subjected to the arbitrary boundary condition. The compliance properties of the beam are assumed to vary linearly along the axis of the beam. By using the Reissner-type variational principle, governing equations derived for the functionally graded beam in the form of in-plane and out-of-plane coordinate variables. Recently developed multi-term extended Kantorovich method is used to transform governing equations into sets of coupled algebraic-ordinary differential equations along in-plane direction and thickness direction. The coefficient of ordinary differential equations (ODEs) along the thickness has constant coefficients which are solved exactly. But the ODEs along  $x$ -axis have variable coefficients due to the variation of material properties along the axis of the beam. Therefore, along  $x$ -axis, the set of ODEs is solved by employing the modified power series methodology. The numerical results are presented for single and multi-layered beam subjected to different boundary conditions. The

effects of material property variations are investigated by plotting the results for the various cases along with the homogeneous beam. The bending behavior of beam is greatly influenced by small increment in material properties.

## 2 ELASTICITY FORMULATION OF BEAM

### 2.1 Material property variation of beam

The compliances of the layer are assumed to vary linearly along the  $x$ -direction as:

$$\begin{aligned}\bar{s}_{ij}^m &= \bar{s}_{ij}(1 + \delta_1 \zeta) = \bar{s}_{ij} + \hat{s}_{ij} \quad \text{for } j=1,3 \\ \bar{s}_{55}^m &= \bar{s}_{55}(1 + \delta_2 \zeta) = \bar{s}_{55} + \hat{s}_{55}\end{aligned}\quad (1)$$

where  $\zeta$  non-dimensional parameter along  $x$  ( $\zeta = x/a$ ) and  $\delta_1, \delta_2$  are gradation indexes which control the material property variation and can have any arbitrary value. Indexes can take positive or negative value as per application or requirement.

### 2.2 Basic governing equation for beam

A multilayered axially graded laminated beam ( $x = (0, a), z = -h/2, h/2$ ) as shown in Fig.1, is considered for the analysis. The governing equations hold for each layer ( $k$ th layer) having thickness  $t^{(k)}$ . Since width is very small along  $y$ -direction, the displacement  $u$  and  $w$  along  $x$ -axis and  $z$ -axis, respectively are considered which are independent of  $y$  axis. Therefore, stresses  $\sigma_y, \tau_{yz}, \tau_{xy}$  becomes zero. Using non-zero strain-displacements relations, the two-dimensional constitutive equations for FGM beam are given as,

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} u_{,x} \\ w_{,z} \\ w_{,x} + u_{,z} \end{bmatrix} = \begin{bmatrix} \bar{s}_{11} + \hat{s}_{11} & \bar{s}_{13} + \hat{s}_{13} & 0 \\ \bar{s}_{13} + \hat{s}_{13} & \bar{s}_{33} & 0 \\ 0 & 0 & \bar{s}_{55} + \hat{s}_{55} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_z \\ \tau_{zx} \end{bmatrix}\quad (2)$$

Using Eq.(2), Reissner-type mixed variational principle for an axially graded beam without body force is expressed as,

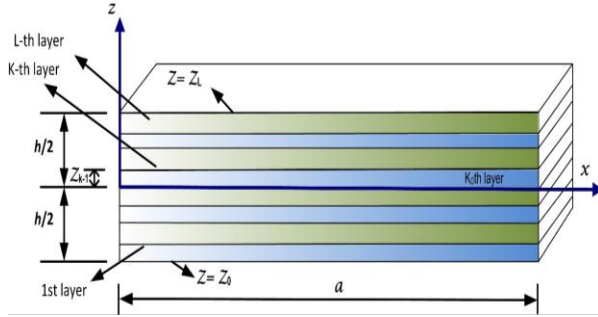
$$\begin{aligned}\int_V [\delta u (\sigma_{x,x} + \tau_{zx,z}) + \delta w (\tau_{zx,x} + \sigma_{z,z}) + \delta \sigma_x \{(\bar{s}_{11} + \hat{s}_{11})\sigma_x + (\bar{s}_{13} + \hat{s}_{13})\sigma_z - u_{,x}\} \\ \delta \sigma_z \{(\bar{s}_{13} + \hat{s}_{13})\sigma_x + \bar{s}_{33}\sigma_z - w_{,z}\} + \delta \tau_{zx} \{(\bar{s}_{55} + \hat{s}_{55})\tau_{zx} - u_{,z} - w_{,x}\}] dV = 0\end{aligned}\quad (3)$$

Beam is subjected to uniform distributed pressure ( $\sigma_z = -p_a$ ) at top and bottom surface and there is no shear stress ( $\tau_{zx}$ ) at bottom and top surface of beam. For perfect inter-laminar bounding case, displacements ( $u, w$ ) and transverse stresses ( $\sigma_z, \tau_{zx}$ ) need to satisfy following condition at the  $k$ th interface

$$\left[ (u, w, \sigma_z, \tau_{zx}) \Big|_{\zeta=1} \right]^{(k)} = \left[ (u, w, \sigma_z, \tau_{zx}) \Big|_{\zeta=0} \right]^{(k+1)}\quad (4)$$

where  $\zeta^{(k)}$  is non-dimensional local thickness parameter for the  $k$ th layer ( $\zeta^{(k)} = (z - z_{k-1})/t^{(k)}$ ) which takes value 0 to 1 for each layer. Along  $x$ -axis, FG beam can have any type of support such as:

- (i) Simply supported ( $\sigma_x = w = 0$ ),
- (ii) Clamped ( $u = w = 0$ ),
- (iii) Free ( $\sigma_x = \tau_{xz} = 0$ ).



**Fig.1**  
Geometry of the FGM laminated beam.

### 3 GENERALIZED MULTI-TERM EKM SOLUTION

There are five primary field variables ( $u, w, \sigma_x, \sigma_z, \tau_{zx}$ ) which are to be solved. Using the multi-term EKM [39], the field variables for the  $k$ th layer are expressed as,

$$[u \ w \ \sigma_x \ \sigma_z \ \tau_{zx}]^T = \sum_{i=1}^n [f_1^i g_1^i \ f_2^i g_2^i \ f_3^i g_3^i \ f_4^i g_4^i + [p_a + zp_d] \ f_5^i g_5^i]^T \tag{5}$$

where  $f$  and  $g$  are unknown functions of  $\zeta$  and  $\xi$ , respectively, and  $p_a = -(p_1 + p_2)/2$  and  $p_d = -(p_2 - p_1)/h$ . The functions  $g_1^i(\xi)$  is dependent on  $k$ th layer, while functions  $f_1^i(\xi)$  are valid for all layers.

#### 3.1 First step - along thickness direction

Functions  $f_1^i(\xi)$  along  $\xi$  direction are assumed in trigonometric form ( $\cos i\pi\xi$  or  $\sin i\pi\xi$ ) which identically satisfy the simply supported boundary conditions. But in multi-term EKM solution, the starting guess functions need not to be assumed as per the boundary conditions. The functions  $g_1^i(\xi)$  are to be solved in this step. For this purpose, first variation  $\delta X_l$  is obtained as,

$$[\delta u \ \delta w \ \delta \sigma_x \ \delta \sigma_z \ \delta \tau_{zx}]^T = \sum_{i=1}^n [f_1^i \delta g_1^i \ f_2^i \delta g_2^i \ f_3^i \delta g_3^i \ f_4^i \delta g_4^i \ f_5^i \delta g_5^i]^T \tag{6}$$

Substitute Eqs. (5) and (6) into Eq. (3) and integration is performed along  $x$ -axis. Since variation is arbitrary, the coefficient of  $\delta g_l^i$  must vanish, yields the following set of governing equation

$$MG_{,\zeta} = \bar{A}^m \bar{G} + \hat{A}^m \hat{G} + \bar{Q}_p^m \tag{7}$$

$$K^m \hat{G} = \tilde{A}^m \bar{G} + \tilde{Q}_p^m \tag{8}$$

where  $M, \bar{A}^m, \hat{A}^m, K^m$  and  $\tilde{A}^m$  are  $4n \times 4n, 4n \times 4n, 4n \times 1n, 1n \times 1n$  and  $1n \times 4n$  matrices.  $\bar{G} = [g_1^1 \dots g_1^n \ g_2^1 \dots g_2^n \ g_4^1 \dots g_4^n \ g_5^1 \dots g_5^n]^T, \hat{G} = [g_3^1 \dots g_3^n]^T$ . Non-zero elements of the matrices are given below:

$$\begin{aligned}
M_{i_1j_1} &= M_{i_4j_4} = \langle f_5^i f_1^j \rangle_a, & M_{i_2j_2} &= M_{i_3j_3} = \langle f_4^i f_2^j \rangle_a \\
\bar{A}_{i_1j_2}^m &= \frac{-t}{a} \langle f_5^i f_{2,\xi}^j \rangle_a, & \bar{A}_{i_1j_4}^m &= t \bar{s}_{55} \langle f_5^i f_5^j \rangle_a + \delta_2 t \bar{s}_{55} \langle \xi f_5^i f_5^j \rangle_a \\
\bar{A}_{i_2j_3}^m &= t \bar{s}_{33} \langle f_4^i f_4^j \rangle_a, & \hat{A}_{i_2j_1}^m &= t \bar{s}_{13} \langle f_4^i f_3^j \rangle_a + \delta_1 t \bar{s}_{13} \langle \xi f_4^i f_3^j \rangle_a \\
\hat{A}_{i_3j_4}^m &= \frac{-t}{a} \langle f_2^i f_{5,\xi}^j \rangle_a, & \hat{A}_{i_4j_1}^m &= \frac{-t}{a} \langle f_1^i f_{3,\xi}^j \rangle_a \\
K_{i_1j_1}^m &= \bar{s}_{11} \langle f_3^i f_3^j \rangle_a + \delta_1 \bar{s}_{11} \langle \xi f_3^i f_3^j \rangle_a, & \tilde{A}_{i_1j_1}^m &= \frac{1}{a} \langle f_3^i f_{1,\xi}^j \rangle_a \\
\tilde{A}_{i_1j_3}^m &= -\bar{s}_{13} \langle f_3^i f_4^j \rangle_a - \delta_1 \bar{s}_{13} \langle \xi f_3^i f_4^j \rangle_a
\end{aligned} \tag{9}$$

where  $i_p = (p-1)n + i$  and  $j_q = (q-1)n + j$  for  $p, q = 1, 2, \dots, 5$ .  $\bar{Q}_p^m, \tilde{Q}_p^m$  are load vectors of size  $4n$ , and  $1n$ , respectively, whose non-zero terms are given by

$$\begin{aligned}
\bar{Q}_{p_{i_3}}^m &= -t \langle f_2^i \rangle_a p_d, & \bar{Q}_{p_{i_2}}^m &= t \bar{s}_{33} \langle f_4^i \rangle_a (p_a^k + \zeta t p_d) \\
\tilde{Q}_{p_{i_2}}^m &= -\bar{s}_{13} \langle f_3^i \rangle_a (p_a^k + \zeta t p_d) - \delta_1 \bar{s}_{13} \langle \xi f_3^i \rangle_a (p_a^k + \zeta t p_d)
\end{aligned} \tag{10}$$

where  $\langle \dots \rangle_a = a \int_0^1 (\dots) d\xi$ , and  $p_a^k = p_a + p_d z_k$ . The functions  $f_l^i$  are known so the above elements of matrices are obtained in closed form. Substituting algebraic Eqs. (8) into Eq. (7) gives:

$$\bar{G}_{,\xi} = A \bar{G} + Q_p \tag{11}$$

with  $A = M^{-1} [\bar{A}^m + \hat{A}^m K^{m-1} \tilde{A}^m]$  and  $Q_p = M^{-1} [\bar{Q}_p^m + \hat{A}^m K^{m-1} \tilde{Q}_p^m]$ . Eq. (11) is a system of simultaneous nonhomogeneous first order differential equations of size  $4n$  which is solved using the technique given by Kapuria and Kumari [39].

### 3.2 Second step - along direction (x)

Now  $g_l^i(\xi)$  is known from the first step (Sec. 3.1) and arbitrary variation is considered along the  $x$ -direction therefore variation for this case is written as:

$$[\delta u \quad \delta w \quad \delta \sigma_x \quad \delta \sigma_z \quad \delta \tau_{zx}]^T = \sum_{i=1}^n [g_1^i \delta f_1^i \quad g_2^i \delta f_2^i \quad g_3^i \delta f_3^i \quad g_4^i \delta f_4^i \quad g_5^i \delta f_5^i]^T \tag{12}$$

Similarly like first step, the functions  $f_l^i(\xi)$  are partitioned into  $\bar{F} = [f_1^1 \dots f_1^n \quad f_2^1 \dots f_2^n \quad f_3^1 \dots f_3^n \quad f_4^1 \dots f_4^n \quad f_5^1 \dots f_5^n]^T$  and  $\hat{F} = [f_4^1 \dots f_4^n]^T$ . Further substituting Eq. (12) in Eq. (3), and integrating along  $z$ -axis, yields the following set of governing equations:

$$N \bar{F}_{,\xi} = \bar{B}^f(\xi) \bar{F} + \hat{B}^f(\xi) \hat{F} + \bar{P}_m^f(\xi) \tag{13}$$

$$L \hat{F} = \tilde{B}^f(\xi) \bar{F} + \tilde{P}_m^f \tag{14}$$

where  $N$ ,  $\bar{B}^f$ ,  $\hat{B}^f$ ,  $L$  and  $\tilde{B}^f$  are  $4n \times 4n$ ,  $4n \times 4n$ ,  $4n \times 1n$ ,  $1n \times 1n$  and  $1n \times 4n$  matrices respectively, and  $\bar{P}_m^f$  and  $\tilde{P}_m^f$  are  $4n \times 1$  and  $1n \times 1$  load column vectors. The nonzero terms of these matrices are given as:

$$\begin{aligned}
 N_{i_1j_1} &= N_{i_3j_3} = \langle g_3^i g_1^j \rangle_h & N_{i_2j_2} &= N_{i_4j_4} = \langle g_5^i g_2^j \rangle_h \\
 \bar{B}_{i_1j_3}^f &= (1 + \delta_1 \xi) \langle \bar{s}_{11} g_3^i g_3^j \rangle_h & \hat{B}_{i_1j_1}^f &= (1 + \delta_1 \xi) \langle \bar{s}_{13} g_3^i g_4^j \rangle_h \\
 \bar{B}_{i_2j_1}^f &= -\frac{1}{t} \langle g_5^i g_{1,\zeta}^j \rangle_h & \bar{B}_{i_2j_4}^f &= (1 + \delta_2 \xi) \langle \bar{s}_{55} g_5^i g_5^j \rangle_h \\
 \bar{B}_{i_3j_4}^f &= \frac{1}{t} \langle g_5^i g_{1,\zeta}^j \rangle_h & \hat{B}_{i_4j_1}^f &= -\frac{1}{t} \langle g_2^i g_{4,\zeta}^j \rangle_h \\
 L_{i_1j_1} &= \langle \bar{s}_{33} g_4^i g_4^j \rangle_h & \tilde{B}_{i_1j_3}^f &= -(1 + \delta_1 \xi) \langle \bar{s}_{13} g_4^i g_3^j \rangle_h \\
 \tilde{B}_{i_1j_2}^f &= \frac{1}{t} \langle g_4^i g_{2,\zeta}^j \rangle_h & \bar{P}_{m_{i1}}^f &= (1 + \delta_1 \xi) \langle \bar{s}_{13} g_3^i (p_a^k + \zeta t p_d) \rangle_h \\
 \bar{P}_{m_{i4}}^f &= -p_d \langle g_2^i \rangle_h & \tilde{P}_{m_{i1}}^f &= -\langle \bar{s}_{33} g_4^i (p_a^k + \zeta t p_d) \rangle_h
 \end{aligned} \tag{15}$$

where  $\langle \dots \rangle_h = \sum_{k=1}^L t^{(k)} \int_0^1 (\dots)^{(k)} d\zeta$ , . Since functions  $g_i^j(\zeta)$  are known in close form from previous step, so the above elements of matrices are obtained in closed form. Substituting algebraic Eqs. (14) into Eq. (13) gives:

$$\bar{F}_{,\xi} = B(\xi) \bar{F} + P(\xi) \tag{16}$$

where  $B = N^{-1}[\bar{B}^f + \hat{B}^f L^{-1} \tilde{B}^f]$ ;  $P = N^{-1}[\bar{P}_m^f + \hat{B}^f L^{-1} \tilde{P}_m^f]$ .

### 3.3 Solution technique

Eq. (16) is a system of simultaneous nonhomogeneous first order differential equations ( $4n$ ) with variable coefficients which is solved using modified power series technique given by Kumari et al. [41]. Thus, general solution is approximated as:

$$\bar{F}^j(\xi) = \sum_{i=0}^{N_p} Z_i^j \xi^i + \left( \sum_{i=0}^{N_p} H_i^j \xi^i \right) C_0 \tag{17}$$

where constants  $Z_i$ ,  $H_i$  obtained from recursive relations and unknown constant  $C_0$  is calculated by applying  $x$ -direction boundary conditions.  $N_p$  is the number of terms in power series which is chosen large enough so that the contribution of further terms is negligible and less than a stipulated small number  $\eta$  ( $= 10^{-10}$ ). For detailed solution procedure one can refer article Ref. [41]. Now  $\bar{F}$  is known function which is substituted into Eq. (14) to solve the function  $\hat{F}$ . Now the second step is completed and further, these two steps of thickness and in-plane directions are repeated to achieve the required level of accuracy.

## 4 NUMERICAL RESULTS AND DISCUSSIONS

Numerical results are presented and discussed for a single layer and multi-layered beams ( $a = 1, S = 5$ ). Beams are subjected to a uniformly distributed pressure load  $p_2 = p_0$  at the top surface. The material properties of beam are

taken as:  $[(Y_1, Y_2, Y_3, G_{23}, G_{13}, G_{12}), \nu_{12}, \nu_{13}, \nu_{23}] = [(181.0, 10.3, 10.3, 2.87, 7.17, 7.17) \text{ GPa}, 0.28, 0.28, 0.33]$ . The present results are non-dimensionalized with  $S = a/h$ ,  $Y_0 = 10.3 \text{ GPa}$  and  $p_0 = 1$  as:  $(\bar{u}, \bar{w}) = 100(u, w / S Y_0 / p_0 h S^3)$ ;

$(\bar{\sigma}_x, \bar{\tau}_{zx}) = (\sigma_x, \tau_{zx}) / p_0 S^2$ . The beams are designated according to their support conditions at the edges  $\xi = 0, 1$ . For example, the beam which is clamped (*C*) at  $\xi = 0$  and free (*F*) at  $\xi = 1$ , is called an *C-F* beam. For axially graded beam, there are no analytical results available in the literature. Therefore, accuracy and efficiency of the present EKM method are verified thoroughly by comparing with 2D FE. Since it is a beam with a small width, plane stress element of ABAQUS [43] is used. Therefore, the 2D plane beam, with length  $a$  along  $x$ -direction and thickness  $h$  along  $z$ -direction is modeled in ABAQUS using the element type CPS8R with a mesh size of 50 (length)  $\times$  16 (thickness). For FG beams, the spatially graded property distribution (at different Gauss points) is implemented by employing user material subroutine (UMAT) [41]. Converged results of EKM are presented for all the cases.

#### 4.1 Validation

A simply-supported composite beam consisting of four-ply of equal thickness  $0.25h$  is considered for the validation. The accuracy of the present method has been investigated by comparing the result for beams with symmetric lay-up  $[0^\circ/90^\circ/90^\circ/0^\circ]$  and unsymmetrical lay-up  $[0^\circ/90^\circ/0^\circ/90^\circ]$  as listed in Table 1. The present results are in excellent agreement with the results are given in Ref. [42] for the simply-supported thick beam ( $S = 5$ ), moderate beam ( $S = 10$ ) and thin beam ( $S = 20$ ). It is observed that for various span-to-thickness ratios  $S$ , the one-term solution ( $n = 1$ ,  $iter = 1$ ) gives an accurate prediction for simply supported support condition.

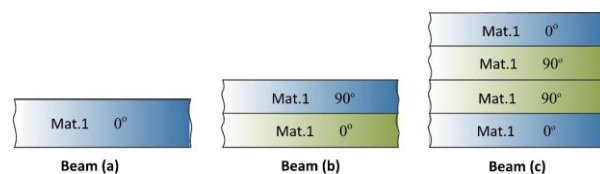
**Table 1**

Comparison of present EKM results with exact 2D elasticity results for homogeneous beam.

Entity	$S$	Lay-up $[0^\circ/90^\circ/90^\circ/0^\circ]$			Lay-up $[90^\circ/0^\circ/90^\circ/0^\circ]$		
		2D Exact [42]	ZIGT [42]	Present	2D Exact [42]	ZIGT [42]	Present
$\bar{w}$ (0.5a,0)	5	-2.6748	-2.6828	-2.6657	-3.7943	-3.7374	-3.7859
	10	-1.4343	-1.4357	-1.4339	-2.4461	-2.4265	-2.4456
	20	-1.1152	-1.1152	-1.1152	-2.0958	-2.0895	-2.0959
$\bar{\sigma}_x$ (0.5a, -0.5h)	5	1.0711	1.0732	1.0995	0.15824	0.14273	0.16365
	10	0.90587	0.90587	0.91728	0.12679	0.12273	0.12895
	20	0.86414	0.86414	0.86731	0.11878	0.11783	0.11939
$\bar{\sigma}_x$ (0.5a, 0.5h)	5	-1.0602	-1.0729	-1.1037	-1.4862	-1.5011	-1.5239
	10	-0.9031	-0.90581	-0.91482	-1.3442	-1.3469	-1.3546
	20	-0.8635	-0.86436	-0.86646	-1.3081	-1.3094	-1.3108

#### 4.2 Effect of various parameter on single layered beam (a)

Single-layer thick beam (a) (as shown in Fig. 2) is considered for study in this section.



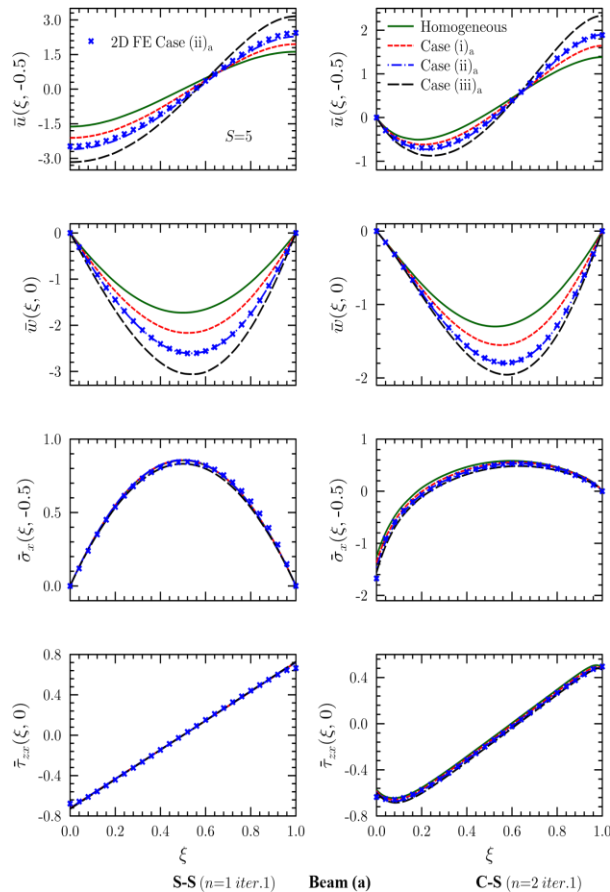
**Fig.2**  
Configuration of FGM beams.

Longitudinal variation of deflections  $(\bar{u}, \bar{w})$  and stresses  $(\bar{\sigma}_x, \bar{\tau}_{zx})$  for the beam (a) are plotted in Figs. 3, 4 for various boundary conditions. Results are plotted for three different variation cases  $\{Case(i)_a: \delta_1 = \delta_2 = 0.5, Case(ii)_a: \delta_1 = \delta_2 = 1.0, \text{ and } Case(iii)_a: \delta_1 = 2.0, \delta_2 = 1.0\}$  along with homogeneous case to investigate the effect of axially varying material properties on bending behavior of the beam. 2D FE results for case  $(ii)_a$  are also plotted in these

figures. It is observed that present EKM results are in excellent agreement with 2D FE results except for stresses at very clamped edges. This mismatch near the clamped edge is because the FE solution does not satisfy the conditions of applied normal and transverse stresses ( $\bar{\sigma}_z$  and  $\bar{\tau}_{zx}$ ) [39] and it can be verified from Fig. 5. It is observed that for simply-supported boundary condition, the single term ( $n=1, iter.1$ ) gives an accurate prediction for all the entities whereas two-term solution ( $n=2, iter.2$ ) is required for the other type of boundary conditions. It is evident that for all the boundary conditions (S-S, C-S, C-F, C-C), deflections ( $\bar{u}, \bar{w}$ ) are effected significantly with increase in variation indices whereas stresses ( $\bar{\sigma}_x, \bar{\tau}_{zx}$ ) have almost no effect.

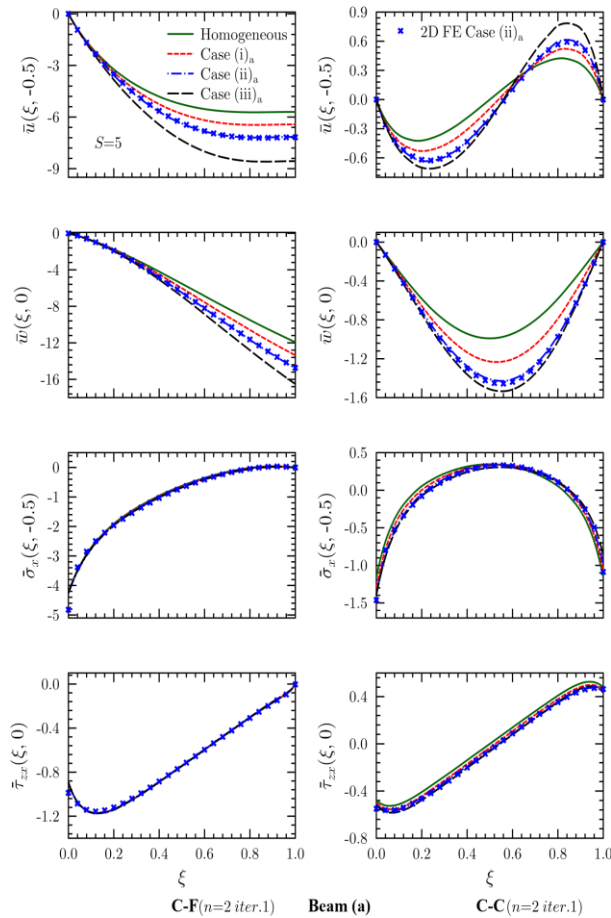
As variation indices increases, the point of maximum deflection ( $\bar{w}$ ) for S-S, C-S, and C-C boundary conditions is shifted gradually toward  $\xi = 1.0$ . For S-S and C-F support conditions, percentage (%) increment in  $\bar{w}$  is nearly equal for all the cases. But percentage increment in  $\bar{w}$  for Case ( $iii_a$ ) is very less as compared to Case ( $i_a$ ) and Case ( $ii_a$ ) for C-S and C-C support conditions.

For all the support conditions, axial displacement ( $\bar{u}$ ) is influenced profoundly by an increase in variation index. It is observed that effect of varying material property on  $\bar{u}$  is more pronounced near to simply-supported and free edges than the center of the beam. For S-S, C-S and C-C conditions, the effect of gradation on are minimum (around  $\xi = 0.6$ ) with respect to the constant case. All the lines cut the homogeneous case line at one point near to  $\xi = 0.6$ . For all the boundary conditions, the axial displacement ( $\bar{u}$ ) increases abruptly for Case ( $iii_a$ ) as compared to Case ( $i_a$ ) and Case ( $ii_a$ ). The in-plane displacement ( $\bar{u}$ ) becomes asymmetrical as the variation index increases for S-S and C-C boundary conditions. Asymmetry in the longitudinal variation is maximum for Case ( $iii_a$ ) for both C-C and S-S conditions, and similar trends are observed for C-S case also. Fig. 5 shows the longitudinal variation of normal stress  $\bar{\sigma}_z$  for case ( $i_a$ ) along with homogeneous beam ( $a$ ) for different boundary conditions. For C-C and C-S conditions, a longitudinal variation of  $\bar{\sigma}_z$  becomes asymmetric for the axially graded beam while symmetric homogeneous beam (which is obvious). Figs. 3, 4 and 5 revealed that influence of axially graded material properties on the behavior of the beam depends significantly on boundary conditions of the beam.

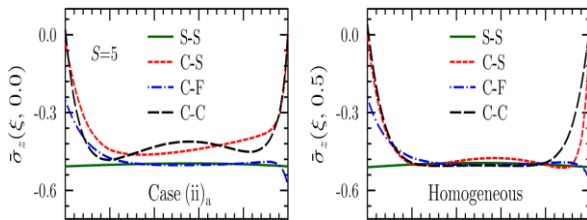


**Fig.3** Effect of variation in properties on longitudinal variations of deflection and stresses for the beam ( $a$ ) ( $S = 5$ ) subjected to S-S and C-S boundary conditions.



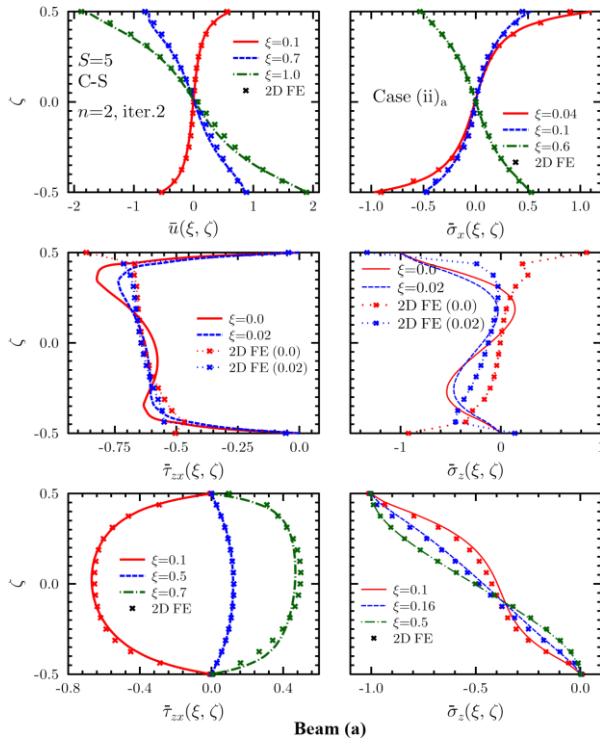


**Fig.4** Effect of variation in properties on longitudinal variations of deflection and stresses for beam (a) ( $S = 5$ ) subjected to C-F and C-C boundary conditions.

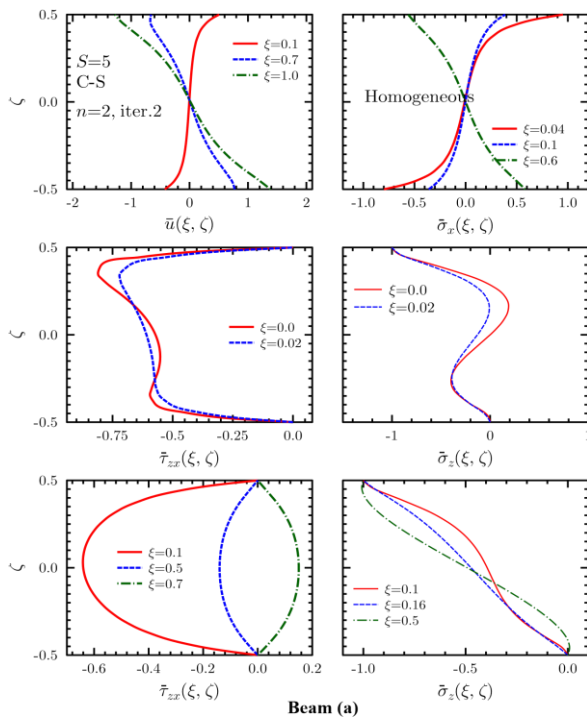


**Fig.5** Longitudinal variations of stress  $\sigma_z$  for homogeneous and FGM beam (b) ( $S = 5$ ).

Through-thickness variations at different  $\xi$  location are depicted in Fig. 6 for Case ( $ii_a$ ) under C-S support conditions. 2D FE results for all  $\xi$  location are also plotted in the Fig. 6. It is observed that the distribution of in-plane displacement  $\bar{u}$  and normal stress  $\bar{\sigma}_x$  match well with the 2D FE solution near to clamped support, but 2D FE does not predict the stresses ( $\bar{\sigma}_z$  and  $\bar{\tau}_{zx}$ ) accurately near the clamped edge. In the vicinity of the clamped edge, 2D FE does not satisfy the boundary conditions for  $\bar{\sigma}_z$  and  $\bar{\tau}_{zx}$  at the top and bottom surfaces of beams whereas present EKM solution satisfies these conditions exactly. As we move away from the clamped edge, 2D FE matches excellently with the present solution. It is observed that in-plane displacement ( $\bar{u}$ ) and stress  $\bar{\sigma}_x$  vary nonlinearly across the thickness near the supports. In-plane displacement ( $\bar{u}$ ) is assumed as a linear function of  $z$  along the thickness of one-dimensional theory based on Euler-Bernoulli and Timoshenko beam theory. Therefore, these theories cannot predict the behavior accurately near the edges. Through-thickness variation of  $\bar{\tau}_{zx}$  is parabolic and its magnitude decreases as we move from clamped edge to center of the beam. Similarly, through-thickness variations at different  $\xi$  location are presented in Fig. 7 for homogeneous beam under C-S support conditions. Similar trends are observed for the homogeneous case also. These benchmark results can be used to verify the accuracy of the 1D solution for homogeneous as well as axially FG beams.



**Fig.6** Comparison of stresses distributions at different  $\xi$ -locations with 2D FE solution for beam (a) (Case (ii)  $a_s$ ,  $S=5$ ) under C-S boundary condition.



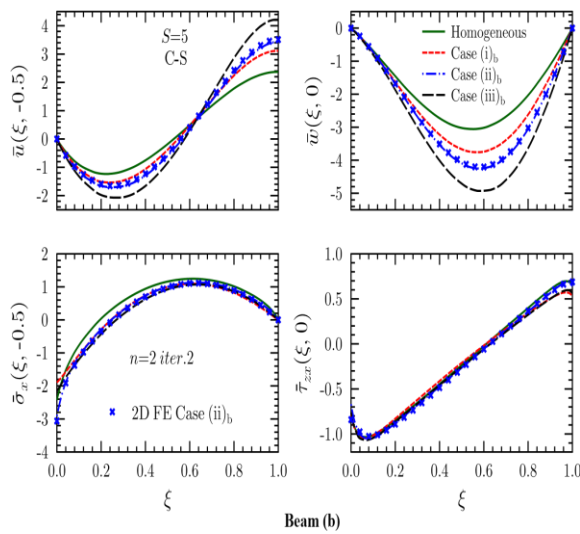
**Fig.7** Comparison of stresses distributions at different  $\xi$ -locations for homogeneous beam (a) ( $S = 5$ ) under C-S boundary condition.

4.3 Effect of various parameter on two layered beam (b)

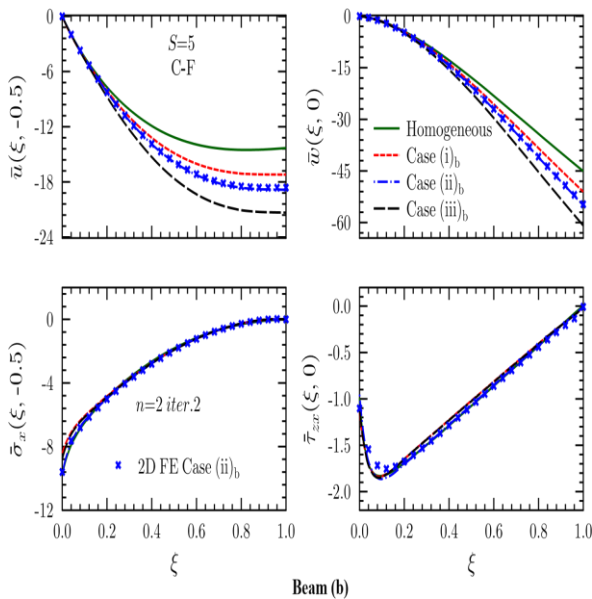
A two-layered beam (b) with the unsymmetrical lay-up schemes of  $[0^0/90^0]$ , as shown in Fig. 2, is considered for analysis in this section. Figs. 8, 9 and 10 depict the effect of variation index on the longitudinal variation of deflections ( $\bar{u}, \bar{w}$ ) and stresses ( $\bar{\sigma}_x, \bar{\tau}_{zx}$ ) of the beam (b) for C-S, C-F, and C-C support conditions, respectively. Beam (b) is a two-layered beam in which effect of longitudinal varying material property is investigated by taking

different and same variation indices ( $\delta_1, \delta_2$ ) for the top and bottom layer. Results are plotted for following variation cases: Case ( $i_b$ ):- Bottom layer  $\delta_1=\delta_2=1.0$ , Top layer  $\delta_1=\delta_2=0.0$ ; Case ( $ii_b$ ):- Bottom layer  $\delta_1=\delta_2=1.0$ , Top layer  $\delta_1=\delta_2=1.0$ ; Case ( $iii_b$ ):- Bottom layer  $\delta_1=\delta_2=2.0$ , Top layer  $\delta_1=\delta_2=1.0$

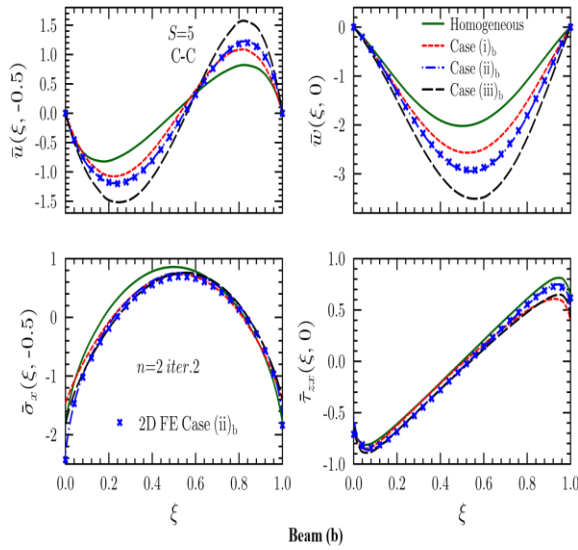
The results for the homogeneous beam (no variation in material properties) are also plotted in Figs. 8, 9 and 10. It is observed that deflections ( $\bar{u}, \bar{w}$ ) are affected significantly by all type of boundary condition even if the axial variation of material property is considered in one layer. Whereas stresses  $\bar{\sigma}_x$  and  $\bar{\tau}_{zx}$  have no significant effect. 2D FE results for case ( $ii_b$ ) are also presented in Figs. 8, 9 and 10. The present results are in excellent agreement with 2D FE except for stress at very clamped edges. Fig. 11 shows the through-thickness distribution of  $\bar{u}$  and stresses ( $\bar{\sigma}_x, \bar{\sigma}_z$  and  $\bar{\tau}_{zx}$ ) for case ( $ii_b$ ) under C-F boundary conditions. It is observed that stresses  $\bar{\sigma}_z$  and  $\bar{\tau}_{zx}$  are highly non-linear near the clamped support and 2 D FE fails to predict it because finite element solution does not satisfy the boundary conditions and interface continuity conditions at the very clamped edge.



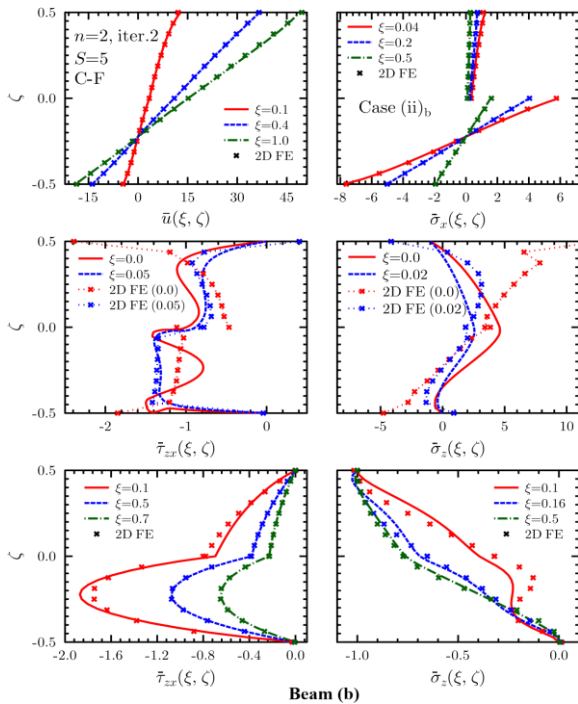
**Fig.8**  
Effect of variation in properties on longitudinal variations of deflection and stresses for beam ( $b$ ) subjected to C-S boundary condition.



**Fig.9**  
Effect of variation in properties on longitudinal variations of deflection and stresses for beam ( $b$ ) subjected to C-F boundary condition.



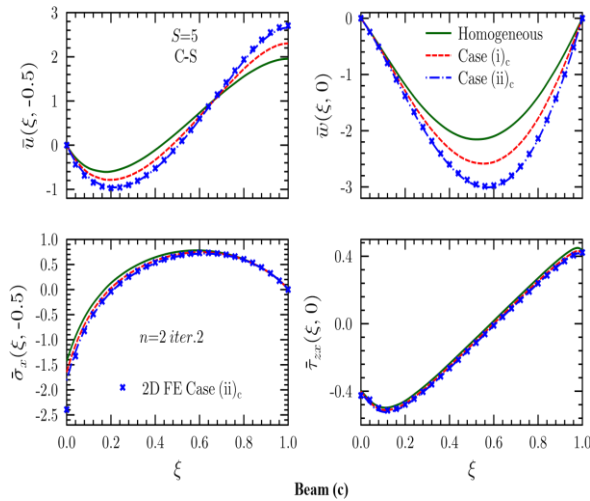
**Fig.10** Effect of variation in properties on longitudinal variations of deflection and stresses for beam (b) ( $S=5$ ) under C-C boundary condition.



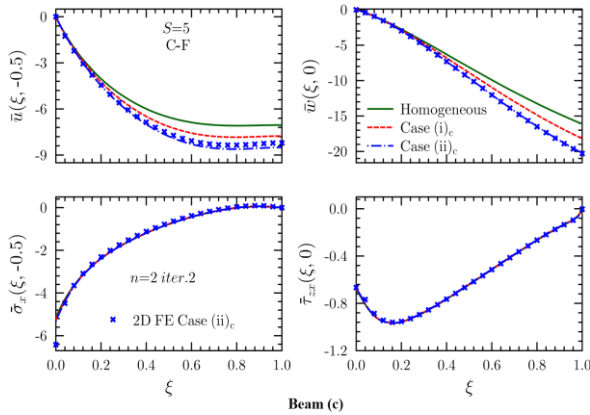
**Fig.11** Comparison of stresses distributions at different  $\xi$ -locations with 2D FE solution for FGM beam (b) ( $Case (ii)_b$ ,  $S=5$ ) under C-F boundary condition.

4.4 Effect of various parameter on four-layer beam (c)

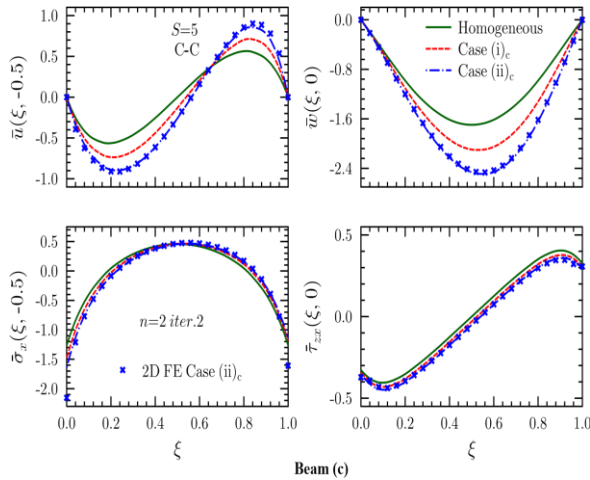
A four-layered beam (c) with the symmetrical lay-up schemes of  $[0^0/90^0/90^0/0^0]$ , as shown in Fig. 2, is analyzed in this section. Results are presented for two cases of property variation, Case (i<sub>c</sub>)  $\delta_1=\delta_2=1.0$  for all layer; Case (ii<sub>c</sub>)  $\delta_1=\delta_2=2.0$  for all layer. Longitudinal variation of deflections ( $\bar{u}, \bar{w}$ ) and stresses ( $\bar{\sigma}_x, \bar{\tau}_{zx}$ ) have been plotted in Figs. 12, 13 and 14 for clamped-simply supported (C-S), clamped-free supported (C-F) and clamped-clamped (C-C) boundary conditions, respectively. Here, results for the layerwise homogeneous beam (c) are also plotted to show the effect of axially graded material properties on deflections and stresses. 2D FE results for Case (ii<sub>c</sub>) are also plotted which are in good agreement with present results except for stresses at very clamped support. It is revealed that the effect of axial gradation of the property is more significant for deflections ( $\bar{u}, \bar{w}$ ) as compared to stresses ( $\bar{\sigma}_x, \bar{\tau}_{zx}$ ) for all three boundary condition (C-S, C-F, C-C). The longitudinal variation of deflections ( $\bar{u}, \bar{w}$ ) become more asymmetric as the variation indexes increase particularly for C-C boundary condition.



**Fig.12**  
Effect of variation in properties on longitudinal variations of deflection and stresses for beam (c) ( $S=5$ ) under C-S boundary condition.



**Fig.13**  
Effect of variation in properties on longitudinal variations of deflection and stresses for beam (c) ( $S=5$ ) under C-F boundary condition.



**Fig.14**  
Effect of variation in properties on longitudinal variations of deflection and stresses for beam (c) ( $S=5$ ) under C-C boundary condition.

**5 CONCLUSIONS**

In this paper, two-dimensional elasticity analytical solution is presented for axially functionally graded beam subjected to the arbitrary boundary condition. Further, 2D elasticity solution for the arbitrary supported homogeneous beam is obtained as a special case of present development. New benchmark results are presented for the axially functionally graded beam as well as a homogeneous beam subjected to arbitrary support conditions. The

influence of material properties variation on the bending response of axially FG beam, as compared to the homogeneous beam, is investigated comprehensively by considering various cases.

Based on the present numerical results, following conclusions are drawn:

- The effects of axial gradation depend significantly on the type of support conditions.
- For S-S and C-C boundary conditions, the longitudinal variation of in-plane displacement ( $\bar{u}$ ) becomes asymmetrical as the variation indices increases.
- For both axially graded and homogeneous beams, in-plane displacement ( $\bar{u}$ ) and normal stress ( $\bar{\sigma}_x$ ) vary nonlinearly through the thickness near the edges. Therefore, 1D theories such as Euler-Bernoulli and Timoshenko beam theory cannot predict bending behavior accurately. That is why two-dimensional elasticity solutions are required to predict the bending behavior of beams accurately near to edges.
- The axial gradation of material property even in one layer significantly affects the deflections ( $\bar{u}, \bar{w}$ ) for all boundary condition.
- Normal stress ( $\bar{\sigma}_z$ ) and transverse shear stress ( $\bar{\tau}_{zx}$ ) are highly non-linear near to clamped support for axially graded as well as homogeneous beams. 2D FE fails to predict it because finite element solution does not satisfy the boundary conditions and interface continuity conditions at the very clamped edge.

This development has shown that, by controlling the axial variation of material properties, the desired response of beam can be achieved for specific applications. The current research will also be beneficial to modeled real-life beam structures in which material properties of beam deteriorate due to some environmental effect. The present results may be used for assessing the validity and accuracy of different beam theories and computational models for analysis of axially graded beams. The present formulation can be modified for another type of material property variations.

## REFERENCES

- [1] Nakamura T., Singh R., Vaddadi P., 2006, Effects of environmental degradation on flexural failure strength of fiber reinforced composites, *Experimental Mechanics* **46**(2): 257-268.
- [2] Barbero E., Cosso F., Campo F., 2013, Benchmark solution for degradation of elastic properties due to transverse matrix cracking in laminated composites, *Composite Structures* **98**: 242-252.
- [3] Adda-Bedia E., Bouazza M., Tounsi A., Benzair A., Maachou M., 2008, Prediction of stiffness degradation in hygrothermal aged  $[\theta_m/90_n]_s$  composite laminates with transverse cracking, *Journal of Materials Processing Technology* **199**(1): 199-205.
- [4] Naebe M., Shirvanimoghaddam K., 2016, Functionally graded materials: A review of fabrication and properties, *Applied Materials Today* **5**: 223-245.
- [5] Sankar B., 2001, An elasticity solution for functionally graded beams, *Composites Science and Technology* **61**(5): 689-696.
- [6] Ding H., Huang D., Chen W., 2007, Elasticity solutions for plane anisotropic functionally graded beams, *International Journal of Solids and Structures* **44**(1): 176-196.
- [7] Zhong Z., Yu T., 2007, Analytical solution of a cantilever functionally graded beam, *Composites Science and Technology* **67**(3): 481-488.
- [8] Kapuria S., Bhattacharyya M., Kumar A., 2008, Bending and free vibration response of layered functionally graded beams: A theoretical model and its experimental validation, *Composite Structures* **82**(3): 390-402.
- [9] Kadoli R., Akhtar K., Ganesan N., 2008, Static analysis of functionally graded beams using higher order shear deformation theory, *Applied Mathematical Modelling* **32**(12): 2509-2525.
- [10] Nguyen T.K., Vo T.P., Thai H.T., 2013, Static and free vibration of axially loaded functionally graded beams based on the first-order shear deformation theory, *Composites Part B: Engineering* **55**: 147-157.
- [11] Pradhan K., Chakraverty S., 2014, Effects of different shear deformation theories on free vibration of functionally graded beams, *International Journal of Mechanical Sciences* **82**: 149-160.
- [12] Sallai B., Hadji L., Daouadji T.H., Adda B., 2015, Analytical solution for bending analysis of functionally graded beam, *Steel and Composite Structures* **19**(4): 829-841.
- [13] Filippi M., Carrera E., Zenkour A., 2015, Static analyses of FGM beams by various theories and finite elements, *Composites Part B: Engineering* **72**: 1-9.
- [14] Jing L.I., Ming P.J., Zhang W.P., Fu L.R., Cao Y.P., 2016, Static and free vibration analysis of functionally graded beams by combination Timoshenko theory and finite volume method, *Composite Structures* **138**: 192-213.
- [15] Aldousari S., 2017, Bending analysis of different material distributions of functionally graded beam, *Applied Physics A* **123**(4): 296.
- [16] Ghumare S.M., Sayyad A.S., 2017, A new fifth-order shear and normal deformation theory for static bending and elastic buckling of P-FGM beams, *Latin American Journal of Solids and Structures* **14**: 1893-1911.

- [17] Elishakoff I., Candan S., 2001, Apparently first closed-form solution for vibrating: inhomogeneous beams, *International Journal of Solids and Structures* **38**(19): 3411-3441.
- [18] Huang Y., Li X.F., 2010, A new approach for free vibration of axially functionally graded beams with non-uniform cross-section, *Journal of Sound and Vibration* **329**(11): 2291-2303.
- [19] Giunta G., Belouettar S., Carrera E., 2010, Analysis of FGM beams by means of classical and advanced theories, *Mechanics of Advanced Materials and Structures* **17**(8): 622-635.
- [20] Sarkar K., Ganguli R., 2013, Closed-form solutions for non-uniform Euler-Bernoulli free-free beams, *Journal of Sound and Vibration* **332**(23): 6078-6092.
- [21] Sarkar K., Ganguli R., 2014, Closed-form solutions for axially functionally graded Timoshenko beams having uniform cross-section and fixed-fixed boundary condition, *Composites Part B: Engineering* **58**: 361-370.
- [22] Li X.F., Kang Y.A., Wu J.X., 2013, Exact frequency equations of free vibration of exponentially functionally graded beams, *Applied Acoustics* **74**(3): 413-420.
- [23] Tang A.Y., Wu J.X., Li X.F., Lee K., 2014, Exact frequency equations of free vibration of exponentially non-uniform functionally graded Timoshenko beams, *International Journal of Mechanical Sciences* **89**: 1-11.
- [24] Nguyen N., Kim N., Cho I., Phung Q., Lee J., 2014, Static analysis of transversely or axially functionally graded tapered beams, *Materials Research Innovations* **18**: 260-264.
- [25] Kukla S., Rychlewska J., 2016, An approach for free vibration analysis of axially graded beams, *Journal of Theoretical and Applied Mechanics* **54**(3): 859-870.
- [26] Huang Y., Rong H.W., 2017, Free vibration of axially inhomogeneous beams that are made of functionally graded materials, *International Journal of Acoustics & Vibration* **22**(1): 68-73.
- [27] Shahba A., Attarnejad R., Marvi M.T., Hajilar S., 2011, Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions, *Composites Part B: Engineering* **42**(4): 801-808.
- [28] Shahba A., Attarnejad R., Hajilar S., 2013, A mechanical-based solution for axially functionally graded tapered Euler-Bernoulli beams, *Mechanics of Advanced Materials and Structures* **20**(8): 696-707.
- [29] Shahba A., Rajasekaran S., 2012, Free vibration and stability of tapered Euler-Bernoulli beams made of axially functionally graded materials, *Applied Mathematical Modelling* **36**(7): 3094-3111.
- [30] Li S., Hu J., Zhai C., Xie L., 2013, A unified method for modeling of axially and/or transversally functionally graded beams with variable cross-section profile, *Mechanics Based Design of Structures and Machines* **41**(2): 168-188.
- [31] Arefi M., Rahimi G. H., 2013, Nonlinear analysis of a functionally graded beam with variable thickness, *Scientific Research and Essays* **8**(6): 256-264.
- [32] Giunta G., Belouettar S., Ferreira A., 2016, A static analysis of three-dimensional functionally graded beams by hierarchical modelling and a collocation meshless solution method, *Acta Mechanica* **227**(4): 969-991.
- [33] Arefi M., Zenkour A. M., 2017, Vibration and bending analysis of a sandwich micro-beam with two integrated piezo-magnetic face-sheets, *Composite Structures* **159**: 479-490.
- [34] Arefi M., Zenkour A. M., 2017, Size-dependent vibration and bending analyses of the piezo-magnetic three-layer nano-beams, *Applied Physics A* **123**(3): 202.
- [35] Zenkour A. M., Arefi M., Alshehri N. A., 2017, Size-dependent analysis of a sandwich curved nano-beam integrated with piezo-magnetic face-sheets, *Results in Physics* **7**: 2172-2182.
- [36] Arefi M., Zenkour A. M., 2016, A simplified shear and normal deformations nonlocal theory for bending of functionally graded piezo-magnetic sandwich nano-beams in magneto-thermo-electric environment, *Journal of Sandwich Structures & Materials* **18**(5): 624-651.
- [37] Li X., Li L., Hu Y., Ding Z., Deng W., 2017, Bending, buckling and vibration of axially functionally graded beams based on nonlocal strain gradient theory, *Composite Structures* **165**: 250-265.
- [38] Sayyad A.S., Ghugal Y.M., 2017, Bending, buckling and free vibration of laminated composite and sandwich beams: A critical review of literature, *Composite Structures* **171**: 486-504.
- [39] Kapuria S., Kumari P., 2012, Multi-term extended Kantorovich method for three dimensional elasticity solution of laminated plates, *Journal of Applied Mechanics* **79**(6): 061018.
- [40] Kumari P., Kapuria S., Rajapakse R., 2014, Three-dimensional extended Kantorovich solution for Levy-type rectangular laminated plates with edge effects, *Composite Structures* **107**: 167-176.
- [41] Kumari P., Singh A., Rajapakse R., Kapuria S., 2017, Three-dimensional static analysis of Levy-type functionally graded plate with in-plane stiffness variation, *Composite Structures* **168**: 780-791.
- [42] Kapuria S., Dumir P., Jain N., 2004, Assessment of zigzag theory for static loading, buckling, free and forced response of composite and sandwich beams, *Composite Structures* **64**(3):317-327.
- [43] ABAQUS/STANDARD, 2009, User's Manual, Version: 6.9-1.