Mathematical Analysis and its Contemporary Applications
Volume 4, Issue 2, 2022, 17–34
doi: 10.30495/maca.2022.1946498.1040
ISSN 2716-9898

t-norms over fuzzy ideals (implicative, positive implicative) of BCK-algebras

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Abstract. In this paper, we use the notion of t-norms to introduce fuzzy subalgebras, fuzzy ideals, fuzzy implicative ideals, fuzzy positive implicative ideals in BCK-algebras. Next we clarify the links between them and investigate properties of them. Finally, we consider them under intersection, cartesian product and homomorphisms(image and pre image) and we study related properties.

1. Introduction

In 1966, Imai and Iseki introduced the notion of BCK-algebra [4]. After the introduction of the concept of fuzzy sets by Zadeh [58], several researches were conducted on the generalization of the notion of fuzzy sets. Many authors considered the fuzzification of ideals and subalgebras in BCK-algebras [2, 6, 8, 9, 11, 12, 13, 16, 59]. Triangular norms and conorms are operations which generalize the logical conjunction and logical disjunction to fuzzy logic. The author by using norms, investigated some properties of fuzzy algebraic structures [17]-[56]. In this paper, as using t-norm T, we define fuzzy subalgebras, fuzzy ideals, fuzzy implicative ideals, fuzzy positive implicative ideals in BCK-algebras. Next we investigate them with subalgebras, ideals, implicative ideals, positive implicative ideals in BCK-algebras. Also we investigate them under intersection, cartesian product and homomorphisms(image and pre image) and we study related properties.

2020 *Mathematics Subject Classification.* 11S45, 03E72 , 15A60, 55N45, 51A10.

Key words and phrases. Algebra and orders, theory of fuzzy sets, norms, products and intersections, homomorphisms.

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2. Preliminaries

In this section we cite the fundamental definitions and results that will be used in the sequel. For more details we refer readers to [**1**, **3**, **5**, **7**, **10**, **14**, **15**, **16**, **31**, **35**, **57**].

Definition 2.1. By a BCK-algebra we mean a nonempty set X with a binary operation $*$ and a constant 0 satisfying the axioms:
(1) $((x*y)*(x*z)) \leq (z*y)$,
(2) $(x*(x*y)) \leq y$,
(3) $x \leq x$,
(4) $x \leq y$ and $y \leq x$ imply that $x = y$,
(5) $0 \leq x$
for all $x, y, z \in X$.
A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x*y = 0$. In any BCK-algebra X the following holds:
(6) $x*0 = x$,
(7) $x*y \leq x$,
(8) $(x*y)*z = (x*z)*y$,
(9) $(x*z)*(y*z) \leq x*y$,
(10) $x*(x*(x*y)) = x*y$,
(11) if $x \leq y$, then $x*z \leq y*z$ and $z*y \leq z*x$
for all $x, y, z \in X$.

Definition 2.2. A non-empty subset I of a BCK-algebra X is called subalgebra of X if $x*y \in I$ for all $x, y \in I$.

Definition 2.3. A BCK-algebra X is said to be implicative if $x = x*(y*x)$, for all $x, y \in X$.

Definition 2.4. A BCK-algebra X is said to be positive implicative if $(x*y)*z = (x*z)*(y*z)$ for all $x, y, z \in X$.

Definition 2.5. A non-empty subset I of a BCK-algebra X is called an ideal of X if
(1) $0 \in I$,
(2) $x*y \in I$ and $y \in I$ imply that $x \in I$ for all $x, y \in X$.

Definition 2.6. A non-empty subset I of a BCK-algebra X is called an implicative ideal of X if
(1) $0 \in I$,
(2) $(x*(y*x))*z \in I$ and $z \in I$ imply that $x \in I$ for all $x, y, z \in X$.

Definition 2.7. A non-empty subset I of a BCK-algebra X is called a positive implicative ideal of X if

(1) $0 \in I$,
(2) $(x * y) * z \in I$ and $y * z \in I$ imply that $x * z \in I$ for all $x, y, z \in X$.

Definition 2.8. A mapping $f : X \to Y$ of BCK-algebras is called a homomorphism if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$.

Definition 2.9. Let X be an arbitrary set. A fuzzy subset of X, we mean a function from X into $[0, 1]$. The set of all fuzzy subsets of X is called the $[0, 1]$-power set of X and is denoted $[0, 1]^X$. For a fixed $s \in [0, 1]$, the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is called an upper level of μ.

Definition 2.10. Let φ be a function from set X into set Y such that $\mu \in [0, 1]^X$ and $\nu \in [0, 1]^Y$. For all $x \in X, y \in Y$, we define

$$\varphi(\mu)(y) = \sup\{\mu(x) \mid x \in X, \varphi(x) = y\}$$

and

$$\varphi^{-1}(\nu)(x) = \nu(\varphi(x)).$$

Definition 2.11. A t-norm T is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:
(T1) $T(x, 1) = x$ (neutral element),
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),
(T3) $T(x, y) = T(y, x)$ (commutativity),
(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),
for all $x, y, z \in [0, 1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1, y_1) \geq T(x_2, y_2)$.

Example 2.12. (1) Standard intersection t-norm $T_m(x, y) = \min\{x, y\}$.
(2) Bounded sum t-norm $T_b(x, y) = \max\{0, x + y - 1\}$.
(3) algebraic product t-norm $T_p(x, y) = xy$.
(4) Drastic t-norm

$$T_D(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum t-norm

$$T_{nM}(x, y) = \begin{cases} \min\{x, y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \leq T(x,y) \leq T_{\min}(x,y)$ for all $x,y \in [0,1]$.

We say that T be idempotent if for all $x \in [0,1]$ we have $T(x,x) = x$.

Definition 2.13. Let $\mu, \nu \in [0,1]^X$ and define the intersection of μ and ν is denoted by $\mu \cap \nu \in [0,1]^X$ as

$$(\mu \cap \nu)(x) = T(\mu(x), \nu(x))$$

for all $x \in X$.

Definition 2.14. Let $\mu \in [0,1]^X$ and $\nu \in [0,1]^Y$. Define the cartesian product of μ and ν is denoted by $\mu \times \nu \in [0,1]^{X \times Y}$ as

$$(\mu \times \nu)(x,y) = T(\mu(x), \nu(y))$$

for all $(x,y) \in X \times Y$.

Lemma 2.1. *Let T be a t-norm. Then*

$$T(T(x,y), T(w,z)) = T(T(x,w), T(y,z))$$

for all $x,y,w,z \in [0,1]$.

3. Fuzzy subalgebras, ideals, positive implicative ideals of BCK-algebra under t-norms

Throughout this paper, X, Y always mean two BCK-algebras unless otherwise specified.

Definition 3.1. $\mu \in [0,1]^X$ is called a fuzzy subalgebra of X under t-norm T if

$$\mu(x * y) \geq T(\mu_A(x), \mu_A(y))$$

for all $x,y \in X$. Denote by $FST(X)$, the set of all fuzzy subalgebras of X under t-norm T.

Example 3.2. Let $X = \{0,a,b,c\}$ be a set given by the following Cayley table:

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Then $(X,*,0)$ is a BCK-algebra. Define the fuzzy subset $\mu : (X,*,0) \to [0,1]$ as

$$\mu(x) = \begin{cases} 0.35 & \text{if } x = 0,a,c \\ 0.25 & \text{if } x = b \end{cases}$$

Let $T(a,b) = T_p(a,b) = ab$, for all $a,b \in [0,1]$ then $\mu \in FST(X)$.

Proposition 3.1. *Let $\mu \in [0,1]^X$ such that T be idempotent. Then $\mu \in FST(X)$ if and only if the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is either empty or a subalgebra of X for every $s \in [0,1]$.*

Proof. Let $\mu \in FST(X)$ and $x, y \in \mu_s$. Then

$$\mu(x * y) \geq T(\mu(x), \mu(y)) \geq T(s, s) = s$$

thus $x * y \in \mu_s$ and so μ_s will be a subalgebra of X for every $s \in [0,1]$.
Conversely, let μ_s is either empty or a subalgebra of X for every $t \in [0,1]$. Let $s = T(\mu(x), \mu(y))$ and $x, y \in \mu_s$. As μ_s is a subalgebra of X so $x * y \in \mu_s$ and thus

$$\mu(x * y) \geq s = T(\mu(x), \mu(y))$$

so $\mu \in FST(X)$. □

Proposition 3.2. *Let $\mu \in FST(X)$ and T be idempotent. Then $\mu(0) \geq \mu(x)$ for all $x \in X$.*

Proof. Let $x \in X$. Then

$$\mu(0) = \mu(x * x) \geq T(\mu(x), \mu(x)) = \mu(x).$$

Thus $\mu(0) \geq \mu(x)$. □

Definition 3.3. Define $\mu \in [0,1]^X$ is a fuzzy ideal of X under t-norm T if it satisfies the following inequalities:
(1) $\mu(0) \geq \mu(x)$,
(2) $\mu(x) \geq T(\mu(x * y), \mu(y))$,
for all $x, y \in X$.
Denote by $FIT(X)$, the set of all fuzzy ideals of X under t-norm T.

Example 3.4. Let $X = \{0, 1, 2, 3, 4\}$ be a set given by the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Then $(X, *, 0)$ is a BCK-algebra. Define $\mu \in [0,1]^X$ as

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, 2 \\ t & \text{if } x = 1, 3, 4 \end{cases}$$

such that $t \in [0,1]$. Let $T(a,b) = T_p(a,b) = ab$, for all $a, b \in [0,1]$, then $\mu \in FIT(X)$.

Proposition 3.3. *Let $\mu \in [0,1]^X$ and T be idempotent. Then $\mu \in FIT(X)$ if and only if the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is either empty or an ideal of X, for every $s \in [0,1]$.*

Proof. Let $\mu \in FIT(X)$ and $x, y \in X$. Then $\mu(0) \geq \mu(x) \geq s$ and then $0 \in \mu_s$. Also let $x * y \in \mu_s$ and $y \in \mu_s$. Then

$$\mu(x) \geq T(\mu(x * y), \mu(y)) \geq T(s, s) = s$$

thus $x \in \mu_s$. Then μ_s will be an ideal of X for every $s \in [0, 1]$.
Conversely, let μ_s is either empty or an ideal of X for every $s \in [0, 1]$. Let $s = T(\mu(x * y), \mu(y))$ with $x * y \in \mu_s$ and $y \in \mu_s$. Then $x \in \mu_s$ thus

$$\mu(x) \geq s = T(\mu(x * y), \mu(y))$$

so $\mu \in FIT(X)$. □

Proposition 3.4. *Let $\mu \in FIT(X)$ and $x * y \leq z$. Then $\mu(x) \geq T(\mu(y), \mu(z))$ for all $x, y, z \in X$.*

Proof. As $x * y \leq z$ so $(x * y) * z = 0$ for all $x, y, z \in X$. Then

$$\mu(x) \geq T(\mu(x * y), \mu(y)) \geq T(T(\mu((x * y) * z), \mu(z)), \mu(y))$$

$$= T(T(\mu(0), \mu(z)), \mu(y)) = T(\mu(z), \mu(y)) = T(\mu(y), \mu(z))$$

thus $\mu(x) \geq T(\mu(y), \mu(z))$. □

Proposition 3.5. *Let $\mu \in FIT(X)$ and $x \leq y$ for all $x, y \in X$. Then $\mu(x) \geq \mu(y)$.*

Proof. Since $x \leq y$ so $x * y = 0$ for all $x, y \in X$. Then

$$\mu(x) \geq T(\mu(x * y), \mu(y)) = T(\mu(0), \mu(y)) = \mu(y)$$

therefore $\mu(x) \geq \mu(y)$. □

In the following proposition every $FIT(X)$ is $FST(X)$.

Proposition 3.6. *If $\mu \in FIT(X)$, then $\mu \in FST(X)$.*

Proof. As $x * y \leq x$ so from Proposition 3.9 we get that $\mu(x * y) \geq \mu(x)$. Now

$$\mu(x * y) \geq \mu(x) \geq T(\mu(x * y), \mu(y)) \geq T(\mu(x), \mu(y))$$

and then $\mu \in FST(X)$. □

Remark 3.5. The converse of Proposition 3.10 may not be true. For example in Example 3.2 we have that $\mu \in FST(X)$ but since $\mu(b) = 0.25 \ngeq T(\mu(b * a), \mu(a)) = T(\mu(a), \mu(a)) = \mu(a) = 0.35$ so $\mu \notin FIT(X)$.

Note that under a condition every $FST(X)$ is $FIT(X)$.

Proposition 3.7. *Let $\mu \in FST(X)$. If $\mu(x) \geq T(\mu(y), \mu(z))$ and $x * y \leq z$ for all $x, y, z \in X$, then $\mu \in FIT(X)$.*

Proof. As Proposition 3.4 we get that $\mu(0) \geq \mu(x)$. As $x * (x * y) \leq y$ so $\mu(x) \geq T(\mu(x * y), \mu(y))$. (From the hypothesis)
Then $\mu \in FIT(X)$. □

Definition 3.6. We say that $\mu \in [0,1]^X$ is a fuzzy implicative ideal of X under t-norm T if it satisfies the following inequalities:
(1) $\mu(0) \geq \mu(x)$,
(2) $\mu(x) \geq T(\mu(x * (y * x)), \mu(z))$,
for all $x, y, z \in X$.
Denote by $FIIT(X)$, the set of all fuzzy implicative ideals of X under t-norm T.

Proposition 3.8. *Let $\mu \in [0,1]^X$ and T be idempotent. Then $\mu \in FIIT(X)$ if and only if the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is either empty or an implicative ideal of X for every $s \in [0,1]$.*

Proof. Let $\mu \in FIIT(X)$ and $x, y \in X$. Thus $\mu(0) \geq \mu(x) \geq s$ so $0 \in \mu_s$. Also let $(x * (y * x)) * z \in \mu_s$ and $z \in \mu_s$. Then

$$\mu(x) \geq T(\mu((x * (y * x)) * z), \mu(z)) \geq T(s, s) = s$$

thus $x \in \mu_s$. Then μ_s will be an implicative ideal of X for every $s \in [0,1]$.
Conversely, let μ_s is either empty or an implicative ideal of X for every $s \in [0,1]$. Let $s = T(\mu((x * (y * x)) * z), \mu(z))$ with $(x * (y * x)) * z \in \mu_s$ and $z \in \mu_s$. Then $x \in \mu_s$ thus

$$\mu(x) \geq s = T(\mu((x * (y * x)) * z), \mu(z))$$

so $\mu \in FIIT(X)$. □

Definition 3.7. Define $\mu \in [0,1]^X$ is a fuzzy positive implicative ideal of X under t-norm T if it satisfies the following inequalities:
(1) $\mu(0) \geq \mu(x)$,
(2) $\mu(x * z) \geq T(\mu((x * y) * z), \mu(y * z))$,
for all $x, y, z \in X$.
Denote by $FPIIT(X)$, the set of all fuzzy positive implicative ideals of X under t-norm T.

Proposition 3.9. *Let $\mu \in [0,1]^X$ and T be idempotent. Then $\mu \in FPIIT(X)$ if and only if the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is either empty or a positive implicative ideal of X for every $s \in [0,1]$.*

Proof. Let $\mu \in FPIIT(X)$ and $x, y \in X$. Then $\mu(0) \geq \mu(x) \geq s$ and and then $0 \in \mu_s$.
Also let $(x * y) * z \in A_{s,t}$ and $y * z \in \mu_s$. Then

$$\mu(x * z) \geq T(\mu((x * y) * z), \mu(y * z)) \geq T(s, s) = s$$

thus $x \in \mu_s$. Then μ_s is a posive implicative ideal of X for every $s \in [0,1]$.
Conversely, let μ_s is either empty or a positive implicative ideal of X for every $s \in [0,1]$. Let $s = T(\mu((x*y)*z), \mu(y*z))$ with $(x*y)*z \in \mu_s$ and $y*z \in \mu_s$. Then $x \in \mu_s$ thus

$$\mu(x) \geq s = T(\mu((x*(y*x))*z), \mu(z))$$

so $\mu \in FPIIT(X)$. □

Proposition 3.10. *Let $\mu \in FIT(X)$ such that*

$$\mu(x*y) \geq T(\mu(((x*y)*y)*z), \mu(z))$$

for all $x, y, z \in X$. Then $\mu \in FPIIT(X)$.

Proof. Let $x, y, z \in X$. As properties (8) and (9) of Definition 2.1 we get that

$$((x*z)*z)*(y*z) \leq (x*z)*y = (x*y)*z$$

and from Proposition 3.9 we give that

$$\mu(((x*z)*z)*(y*z)) \geq \mu((x*y)*z).$$

Now by hypothesis if we get $y = z$ and $z = y*z$ we obtain that

$$\mu(x*z) \geq T(\mu(((x*z)*z)*(y*z)), \mu(y*z)).$$

Then

$$\mu(x*z) \geq T(\mu(((x*z)*z)*(y*z)), \mu(y*z)) \geq T(\mu((x*y)*z), \mu(y*z))).$$

Thus $\mu \in FPIIT(X)$. □

Proposition 3.11. *Let $\mu \in FIT(X)$. Then $\mu \in FPIIT(X)$ if and only if*

$$\mu((x*z)*(y*z)) \geq \mu((x*y)*z)$$

for all $x, y, z \in X$.

Proof. Let

$$\mu((x*z)*(y*z)) \geq \mu((x*y)*z)$$

for all $x, y, z \in X$. As properties (9) of Definition 2.1 we get that $(x*z)*(y*z) \leq x*y$ and from Proposition 3.8 we get that

$$\mu(x*z) \geq T(\mu(y*z), \mu(x*y)) \geq T(\mu(y*z), \mu((x*z)*(y*z)))$$
$$= T(\mu((x*z)*(y*z)), \mu(y*z)) \geq T(\mu((x*y)*z), \mu(y*z)).$$

Therefore

$$\mu(x*z) \geq T(\mu((x*y)*z), \mu(y*z))$$

thus $\mu \in FPIIT(X)$.
Conversely, let $\mu \in FPIIT(X)$ and $x, y, z \in X$ with $a = x*(y*z)$ and $b = x*y$. By property (1) of Definition 2.1 we will have that $((x*(y*z))*(x*y)) \leq y*(y*z)$ and thus $((x*(y*z))*(x*y))*z \leq y*(y*z)*z = 0$ (Definition 2.1 property

(1)) and Proposition 3.9 gives us that $\mu(((x*(y*z))*(x*y))*z) \geq \mu(0)$. Then $\mu((a*b)*z) = \mu((x*(y*z)*x*y)*z) \geq \mu(0)$. Now

$$\mu((x*z)*(y*z)) = \mu(x*(y*z)*z) = \mu(a*z) \geq T(\mu((a*b)*z), \mu(b*z))$$
$$\geq T(\mu(0), \mu(b*z)) = \mu(b*z) = \mu((x*y)*z).$$

Thus $\mu((x*z)*(y*z)) \geq \mu((x*y)*z)$ for all $x, y, z \in X$. □

Proposition 3.12. *Let $\mu \in FPIIT(X)$ and $x, y, z, a, b \in X$.*
(1) If $((x*y)*y)*a \leq b$, then

$$\mu(x*y) \geq T(\mu(a), \mu(b)).$$

(2) If $((x*y)*z)*a \leq b$, then

$$\mu((x*z)*(y*z)) \geq T(\mu(a), \mu(b)).$$

Proof. Let $\mu \in FPIIT(X)$ and $x, y, z, a, b \in X$.
(1) Let $((x*y)*y)*a \leq b$ then from Proposition 3.8 we get that $\mu((x*y)*y) \geq T(\mu(a), \mu(b))$. Thus

$$\mu(x*y) \geq T(\mu((x*y)*y), \mu(y*y)) = T(\mu((x*y)*y), \mu(0))$$
$$= \mu((x*y)*y) \geq T(\mu(a), \mu(b))$$

then

$$\mu(x*y) \geq T(\mu(a), \mu(b)).$$

(2) Let $((x*y)*z)*a \leq b$, so from Proposition 3.8 we get that

$$\mu((x*z)*(y*z)) \geq \mu((x*y)*z) \geq T(\mu(a), \mu(b)).$$

□

Proposition 3.13. *Let $\mu \in [0,1]^X$ and $((x*y)*y)*a \leq b$ for all $x, y, a, b \in X$. If $\mu(x*y) \geq T(\mu(a), \mu(b))$, then $\mu \in FPIIT(X)$.*

Proof. First, we prove that $\mu \in FIT(X)$. Let $x, y, z \in X$ such that $x*y \leq z$. Definition 2.1 and Properties (1) give us that $((x*0)*0)*y*z = (x*y)*z = 0$ thus $((x*0)*0)*y \leq z$. Put $y = 0, a = y, b = z$ in hypothesis then $\mu(x) = \mu(x*0) \geq T(\mu(y), \mu(z))$. Thus from Proposition 3.12 we get that $\mu \in FIT(X)$. As $(((x*y)*y)*((x*y)*y))*0 = 0$ so $(((x*y)*y)*((x*y)*y)) \leq 0$ for all $x, y \in X$. Using hypothesis will give us $\mu(x*y) \geq T(\mu((x*y)*y), \mu(0)) = \mu((x*y)*y)$. Therefore $\mu \in FPIIT(X)$. □

Proposition 3.14. *Let $\mu \in [0,1]^X$ and $((x*y)*z)*a \leq b$ for all $x, y, z, a, b \in X$. If*

$$\mu((x*y)*(y*z)) \geq T(\mu(a), \mu(b))$$

then $\mu \in FPIIT(X)$.

PROOF. Let $((x*y)*z)*a \leq b$ for all $x, y, z, a, b \in X$. Then $(((x*y)*z)*a)*b = 0$. Now

$$\mu(x*y) = \mu((x*y)*0) = \mu((x*y)*(y*y)) \geq T(\mu(a), \mu(b))$$

and as Proposition 3.20 we will have that $\mu \in FPIIT(X)$. □

4. Intersection, cartesian product and homomorphism

Proposition 4.1. *Let $\mu, \nu \in FST(X)$. Then $\mu \cap \nu \in FST(X)$.*

PROOF. Let $x, y \in X$. Then

$$(\mu \cap \nu)(x*y) = T(\mu(x*y), \nu(x*y)) \geq T(T(\mu(x), \mu(y)), T(\nu(x), \nu(y)))$$
$$= T(T(\mu(x), \nu(x)), T(\mu(y), \nu(y))) = T((\mu \cap \nu)(x), (\mu \cap \nu)(y))$$

thus

$$(\mu \cap \nu)(x*y) \geq T((\mu \cap \nu)(x), (\mu \cap \nu)(y)).$$

Thus $\mu \cap \nu \in FST(X)$. □

Proposition 4.2. *Let $\mu, \nu \in FIT(X)$. Then $\mu \cap \nu \in FIT(X)$.*

PROOF. Let $x, y \in X$. Then

(1)

$$(\mu \cap \nu)(0) = T(\mu(0), \nu(0)) \geq T(\mu(x), \nu(x)) = (\mu \cap \nu)(x)$$

thus

$$(\mu \cap \nu)(0) \geq (\mu \cap \nu)(x).$$

(2)

$$(\mu \cap \nu)(x) = T(\mu(x), \nu(x)) \geq T(T(\mu(x*y), \mu(y)), T(\nu(x*y), \nu(y)))$$
$$= T(T(\mu(x*y), \nu(x*y)), T(\mu(y), \nu(y))) \text{ (Lemma 2.15)}$$
$$= T((\mu \cap \nu)(x*y), (\mu \cap \nu)(y))$$

so

$$(\mu \cap \nu)(x) \geq T((\mu \cap \nu)(x*y), (\mu \cap \nu)(y)).$$

Then $\mu \cap \nu \in FIT(X)$. □

Proposition 4.3. *If $\mu, \nu \in FIIT(X)$, then $\mu \cap \nu \in FIIT(X)$.*

PROOF. Let $x, y, z \in X$. Then

(1)

$$(\mu \cap \nu)(0) = T(\mu(0), \nu(0)) \geq T(\mu(x), \nu(x)) = (\mu \cap \nu)(x)$$

thus

$$(\mu \cap \nu)(0) \geq (\mu \cap \nu)(x).$$

(2)

$$(\mu \cap \nu)(x) = T(\mu(x), \nu_B(x)) \geq T(T(\mu((x*(y*x))*z), \mu(z)), T(\nu((x*(y*x))*z), \nu(z)))$$

$$= T(T(\mu((x * (y * x)) * z), \nu((x * (y * x)) * z), T(\mu(z), \nu(z))) \text{ (Lemma 2.15)}$$

$$= T((\mu \cap \nu)((x * (y * x)) * z)), (\mu \cap \nu)(z))$$

so

$$(\mu \cap \nu)(x) \geq T((\mu \cap \nu)((x * (y * x)) * z)), (\mu \cap \nu)(z)).$$

Then $\mu \cap \nu \in FIIT(X)$. □

Proposition 4.4. *Let $\mu, \nu \in FPIIT(X)$. Then $\mu \cap \nu \in FPIIT(X)$.*

Proof. Let $x, y, z \in X$. Then

(1)

$$(\mu \cap \nu)(0) = T(\mu(0), \nu(0)) \geq T(\mu(x), \nu(x)) = (\mu \cap \nu)(x)$$

thus

$$(\mu \cap \nu)(0) \geq (\mu \cap \nu)(x).$$

(2)

$$(\mu \cap \nu)(x * z) = T(\mu(x * z), \nu(x * z)) \geq T(T(\mu((x * y) * z), \mu(y * z)), T(\nu((x * y) * z), \nu(y * z)))$$

$$= T(T(\nu((x * y) * z), \nu((x * y) * z)), T(\mu(y * z), \nu(y * z))) \text{ (Lemma 2.15)}$$

$$= T((\mu \cap \nu)((x * y) * z)), (\mu \cap \nu)(y * z))$$

so

$$(\mu \cap \nu)(x * z) \geq T((\mu \cap \nu)((x * y) * z))), (\mu \cap \nu)(y * z)).$$

Therefore $\mu \cap \nu \in FPIIT(X)$. □

Proposition 4.5. *Let $\mu \in FST(X)$ and $\nu \in FST(Y)$. Then $\mu \times \nu \in FST(X \times Y)$.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$(\mu \times \nu)((x_1, y_1) * (x_2, y_2)) = (\mu \times \nu)(x_1 * x_2, y_1 * y_2)$$

$$= T(\mu(x_1 * x_2), \nu(y_1 * y_2)) \geq T(T(\mu(x_1), \mu(x_2)), T(\nu(y_1), \nu(y_2)))$$

$$= T(T(\mu(x_1), \nu)(y_1)), T(\mu(x_2), \nu(y_2))) \text{ (Lemma 2.15)}$$

$$= T((\mu \times \nu)(x_1, y_1), (\mu \times \nu)(x_2, y_2))$$

thus

$$(\mu \times \nu)((x_1, y_1) * (x_2, y_2)) \geq T((\mu \times \nu)(x_1, y_1), (\mu \times \nu)(x_2, y_2)).$$

Therefore $\mu \times \nu \in FST(X \times Y)$. □

Proposition 4.6. *Let $\mu \in FIT(X)$ and $\nu \in FIT(Y)$. Then $\mu \times \nu \in FIT(X \times Y)$.*

Proof. Let $(x,y)\in X\times Y$. Then

$$(\mu\times\nu)(0,0)=T(\mu(0),\nu(0))\geq T(\mu(x),\nu(y))=(\mu\times\nu)(x,y)$$

thus $(\mu\times\nu)(0,0)\geq(\mu\times\nu)(x,y)$.
Also let $x_i\in X$ and $y_i\in Y$ for $i=1,2$. Now

$$(\mu\times\nu)(x_1,y_1)=T(\mu(x_1),\nu(y_1))\geq T(T(\mu(x_1*x_2),\mu(x_2)),T(\nu(y_1*y_2),\nu(y_2)))$$

$$=T(T(\mu(x_1*x_2),\nu(y_1*y_2)),T(\mu(x_2),\nu(y_2)))\ \textit{(Lemma 2.15)}$$

$$=T((\mu\times\nu)(x_1*x_2,y_1*y_2),(\mu\times\nu)(x_2,y_2))=T((\mu\times\nu)((x_1,y_1)*(x_2,y_2)),(\mu\times\nu)(x_2,y_2))$$

thus

$$(\mu\times\nu)(x_1,y_1)\geq T((\mu\times\nu)((x_1,y_1)*(x_2,y_2)),(\mu\times\nu)(x_2,y_2)).$$

Therefore $\mu\times\nu\in FIT(X\times Y)$. □

Proposition 4.7. *Let $\mu\in FIIT(X)$ and $\nu\in FIIT(Y)$. Then $\mu\times\nu\in FIIT(X\times Y)$.*

Proof. Let $(x,y)\in X\times Y$. Then

$$(\mu\times\nu)(0,0)=T(\mu(0),\nu(0))\geq T(\mu(x),\nu(y))=(\mu\times\nu)(x,y).$$

Thus $(\mu\times\nu)(0,0)\geq(\mu\times\nu)(x,y)$.
Also let $x_i\in X$ and $y_i\in Y$ for $i=1,2,3$. Now

$$(\mu\times\nu)(x_1,y_1)=T(\mu(x_1),\nu(y_1))\geq T(T(\mu(x_1*(x_2*x_1)),\mu(x_3)),T(\nu(y_1*(y_2*y_1)),\nu(y_3)))$$

$$=T(T(\mu(x_1*(x_2*x_1)),\nu(y_1*(y_2*y_1))),T(\mu(x_3),\nu(y_3)))\ \textit{(Lemma 2.15)}$$

$$=T((\mu\times\nu)(x_1*(x_2*x_1),y_1*(y_2*y_1)),(\mu\times\nu)(x_3,y_3))$$

$$=T((\mu\times\nu)((x_1,y_1)*((x_2,y_2)*(x_1,y_1)),(\mu\times\nu)(x_3,y_3))$$

thus

$$(\mu\times\nu)(x_1,y_1)\geq T((\mu\times\nu)((x_1,y_1)*((x_2,y_2)*(x_1,y_1)),(\mu\times\nu)(x_3,y_3)).$$

Then $\mu\times\nu\in FIIT(X\times Y)$. □

Proposition 4.8. *Let $\mu\in FPIIT(X)$ and $\nu\in FPIIT(Y)$. Then $\mu\times\nu\in FPIIT(X\times Y)$.*

Proof. Let $(x,y)\in X\times Y$. Then

$$(\mu\times\nu)(0,0)=T(\mu(0),\nu(0))\geq T(\mu(x),\nu(y))=(\mu\times\nu)(x,y)$$

thus $(\mu\times\nu)(0,0)\geq(\mu\times\nu)(x,y)$.
Also let $x_i\in X$ and $y_i\in Y$ for $i=1,2,3$. Then

$$(\mu\times\nu)((x_1,y_1)*(x_3,y_3))=(\mu\times\nu)(x_1*x_3,y_1*y_3)=T(\mu(x_1*x_3),\nu(y_1*y_3))$$

$$\geq T(T(\mu((x_1*x_2)*x_3),\mu(x_2*x_3)),T(\nu((y_1*y_2)*y_3),\nu(y_2*y_3)))$$

$$=T(T(\mu((x_1*x_2)*x_3),\nu((y_1*y_2)*y_3)),T(\mu(x_2*x_3),\nu(y_2*y_3)))\ \textit{(Lemma 2.15)}$$

$$= T((\mu\times\nu)((x_1*x_2)*x_3,(y_1*y_2)*y_3),(\mu\times\nu)(x_2*x_3,y_2*y_3))$$
$$= T((\mu\times\nu)((x_1,y_1)*(x_2,y_2))*(x_3,y_3)),(\mu\times\nu)((x_2,y_2)*(x_3,y_3)))$$

and so

$$(\mu\times\nu)((x_1,y_1)*(x_3,y_3)) \geq T((\mu\times\nu)((x_1,y_1)*(x_2,y_2))*(x_3,y_3)),(\mu\times\nu)((x_2,y_2)*(x_3,y_3))).$$

Then $\mu\times\nu\in FPIIT(X\times Y)$. □

Proposition 4.9. *If $\mu\in FST(X)$ and $\varphi : X\to Y$ be a homomorphism of BCK-algebras, then $\varphi(\mu)\in FST(Y)$.*

Proof. Let $y_1,y_2\in Y$ and $x_1,x_2\in X$ such that $\varphi(x_1)=y_1$ and $\varphi(x_2)=y_2$. Then

$$\varphi(\mu)(y_1*y_2)=\sup\{\mu(x_1*x_2)\mid x_1,x_2\in X,\varphi(x_1)=y_1,\varphi(x_2)=y_2\}$$
$$\geq \sup\{T(\mu(x_1),\mu(x_2)\mid x_1,x_2\in X,\varphi(x_1)=y_1,\varphi(x_2)=y_2\}$$
$$= T(\sup\{\mu(x_1)\mid x_1\in X,\varphi(x_1)=y_1\},\sup\{\mu(x_2)\mid x_2\in X,\varphi(x_2)=y_2\})$$
$$= T(\varphi(\mu)(y_1)),\varphi(\mu)(y_2))$$

thus

$$\varphi(\mu)(y_1*y_2)\geq T(\varphi(\mu)(y_1)),\varphi(\mu)(y_2)).$$

Thus $\varphi(\mu)\in FST(Y)$. □

Proposition 4.10. *If $\nu\in FST(Y)$ and $\varphi : X\to Y$ be a homomorphism of BCK-algebras, then $\varphi^{-1}(\nu)\in FST(X)$.*

Proof. Let $x_1,x_2\in X$. Then

$$\varphi^{-1}(\nu)(x_1*x_2)=\nu(\varphi(x_1*x_2))=\nu(\varphi(x_1)*\varphi(x_2))$$
$$\geq T(\nu(\varphi(x_1)),\nu(\varphi(x_2)))=T(\varphi^{-1}(\nu)(x_1),\varphi^{-1}(\nu)(x_2))$$

thus

$$\varphi^{-1}(\nu)(x_1*x_2)\geq T(\varphi^{-1}(\nu)(x_1),\varphi^{-1}(\nu)(x_2)).$$

Then $\varphi^{-1}(\nu)\in FST(X)$. □

Proposition 4.11. *If $\mu\in FIT(X)$ and $\varphi : X\to Y$ is a homomorphism of BCK-algebras, then $\varphi(\mu)\in FIT(Y)$.*

Proof. Let $x\in X$ and $y\in Y$ with $\varphi(x)=y$. Now

$$\varphi(\mu)(0)=\sup\{\mu(0)\mid 0\in X,\varphi(0)=0\}\geq \sup\{\mu(x)\mid x\in X,\varphi(x)=y\}=\varphi(\mu)(y)$$

thus

$$\varphi(\mu)(0)\geq\varphi(\mu)(y).$$

Also let $x,x_1\in X$ such that $\varphi(x)=y,\varphi(x_1)=y_1$. Then

$$\varphi(\mu)(y)=\sup\{\mu(x)\mid x\in X,\varphi(x)=y\}$$

$$\geq \sup\{T(\mu(x*x_1),\mu(x_1)) \mid x,x_1 \in X, \varphi(x)=y, \varphi(x_1)=y_1\}$$
$$= T(\sup\{\mu(x*x_1) \mid x,x_1 \in X, \varphi(x)=y, \varphi(x_1)=y_1\}, \sup\{\mu(x_1) \mid x_1 \in X, \varphi(x_1)=y_1\})$$
$$= T(\sup\{\mu(x*x_1) \mid x,x_1 \in X, \varphi(x*x_1)=y*y_1\}, \sup\{\mu(x_1) \mid x_1 \in X, \varphi(x_1)=y_1\}$$
$$= T(\varphi(\mu)(y*y_1), \varphi(\mu)(y_1))$$

therefore

$$\varphi(\mu)(y) \geq T(\varphi(\mu)(y*y_1), \varphi(\mu)(y_1)).$$

Thus $\varphi(\mu) \in FIT(Y)$. □

Proposition 4.12. *If $\nu \in FIT(Y)$ and $\varphi : X \to Y$ be a homomorphism of BCK-algebras, then $\varphi^{-1}(\nu) \in FIT(X)$.*

Proof. Let $x \in X$. Then

$$\varphi^{-1}(\nu)(0) = \nu(\varphi(0)) \geq \nu(\varphi(x)) = \varphi^{-1}(\nu)(x).$$

Let $x, x_1 \in X$. As

$$\varphi^{-1}(\nu)(x) = \nu(\varphi(x)) \geq T(\nu(\varphi(x)*\varphi(x_1)), \nu(\varphi(x_1)))$$
$$= T(\nu(\varphi(x*x_1)), \nu(\varphi(x_1))) = T(\varphi^{-1}(\nu)(x*x_1), \varphi^{-1}(\nu)(x_1))$$

so

$$\varphi^{-1}(\nu)(x) \geq T(\varphi^{-1}(\nu)(x*x_1), \varphi^{-1}(\nu)(x_1)).$$

Then $\varphi^{-1}(\nu) \in FIT(X)$. □

Proposition 4.13. *If $\mu \in FIIT(X)$ and $\varphi : X \to Y$ is a homomorphism of BCK-algebras, then $\varphi(\mu) \in FIIT(Y)$.*

Proof. Let $x \in X$ and $y \in Y$ with $\varphi(x) = y$. Now

$$\varphi(\mu)(0) = \sup\{\mu(0) \mid 0 \in X, \varphi(0) = 0\} \geq \sup\{\mu(x) \mid x \in X, \varphi(x) = y\} = \varphi(\mu)(y)$$

thus $\varphi(\mu)(0) \geq \varphi(\mu)(y)$.
Also let $x, x_1, x_2 \in X$ such that $\varphi(x) = y, \varphi(x_1) = y_1, \varphi(x_2) = y_2$. Then

$$\varphi(\mu)(y) = \sup\{\mu(x) \mid x \in X, \varphi(x) = y\}$$
$$\geq \sup\{T(\mu(x*(x_1*x)),\mu(x_2)) \mid x,x_1,x_2 \in X, \varphi(x)=y, \varphi(x_1)=y_1, \varphi(x_2)=y_2\}$$
$$= T(\sup\{\mu(x*(x_1*x)) \mid x,x_1 \in X, \varphi(x)=y, \varphi(x_1)=y_1\}, \sup\{\mu(x_2) \mid x_2 \in X, \varphi(x_2)=y_2\})$$
$$= T(\sup\{\mu(x*(x_1*x)) \mid x,x_1 \in X, \varphi(x*(x_1*x))=y*(y_1*y)\}, \sup\{\mu(x_2) \mid x_2 \in X, \varphi(x_2)=y_2\}$$
$$= T(\varphi(\mu)(y*(y_1*y)), \varphi(\mu)(y_2)).$$

Therefore

$$\varphi(\mu)(y) \geq T(\varphi(\mu)(y*(y_1*y)), \varphi(\mu)(y_2)).$$

Therefore $\varphi(\mu) \in FIIT(Y)$. □

Proposition 4.14. *If $\nu \in FIIT(Y)$ and $\varphi : X \to Y$ be a homomorphism of BCK-algebras, then $\varphi^{-1}(\nu) \in FIIT(X)$.*

PROOF. Let $x \in X$. Then

$$\varphi^{-1}(\nu)(0) = \nu(\varphi(0)) \geq \nu(\varphi(x)) = \varphi^{-1}(\nu)(x)$$

As

$$\varphi^{-1}(\nu)(x) = \nu(\varphi(x)) \geq T(\nu(\varphi(x) * (\varphi(x_1) * \varphi(x)), \nu(\varphi(x_2)))$$

$$= T(\nu(\varphi(x * (x_1 * x))), \nu(\varphi(x_2))) = T(\varphi^{-1}(\nu)(x * (x_1 * x)), \varphi^{-1}(\nu)(x_2))$$

so

$$\varphi^{-1}(\nu)(x) \geq T(\varphi^{-1}(\nu)(x * (x_1 * x)), \varphi^{-1}(\nu)(x_2)).$$

Therefore $\varphi^{-1}(\nu) \in FIIT(X)$. □

Proposition 4.15. *If $\mu \in FPIIT(X)$ and $\varphi : X \to Y$ is a homomorphism of BCK-algebras, then $\varphi(\mu) \in FPIIT(Y)$.*

PROOF. Let $x \in X$ and $y \in Y$ with $\varphi(x) = y$. Now

$$\varphi(\mu)(0) = \sup\{\mu(0) \mid 0 \in X, \varphi(0) = 0\} \geq \sup\{\mu(x) \mid x \in X, \varphi(x) = y\} = \varphi(\mu)(y)$$

thus

$$\varphi(\mu)(0) \geq \varphi(\mu)(y).$$

Also let $x_1, x_2, x_3 \in X$ such that $\varphi(x_1) = y_1, \varphi(x_2) = y_2, \varphi(x_3) = y_3$. Then

$$\varphi(\mu)(y_1 * y_3) = \sup\{\mu(x_1 * x_3) \mid x_1, x_3 \in X, \varphi(x_1) = y_1, \varphi(x_3) = y_3\}$$

$$\geq \sup\{T(\mu((x_1*x_2)*x_3), \mu(x_2*x_3)) \mid x_1, x_2, x_3 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2, \varphi(x_3) = y_3\}$$

$$= T(\sup\{\mu((x_1 * x_2) * x_3)) \mid x_1, x_2, x_3 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2, \varphi(x_3) = y_3\}$$

$$, \sup\{\mu(x_2 * x_3) \mid x_2, x_3 \in X, \varphi(x_2) = y_2, \varphi(x_3) = y_3\})$$

$$= T(\sup\{\mu((x_1 * x_2) * x_3)) \mid x_1, x_2, x_3 \in X, \varphi((x_1 * x_2) * x_3) = (y_1 * y_2) * y_3\}$$

$$, \sup\{\mu(x_2 * x_3) \mid x_2, x_3 \in X, \varphi(x_2 * x_3) = y_2 * y_3\}$$

$$= T(\varphi(\mu)((y_1 * y_2) * y_3), \varphi(\mu_A)(y_2 * y_3))$$

therefore

$$\varphi(\mu)(y_1 * y_3) \geq T(\varphi(\mu)((y_1 * y_2) * y_3), \varphi(\mu)(y_2 * y_3)).$$

Therefore $\varphi(\mu) \in FPIIT(Y)$. □

Proposition 4.16. *If $\nu \in FPIIT(Y)$ and $\varphi : X \to Y$ be a homomorphism of BCK-algebras, then $\varphi^{-1}(\nu) \in FPIIT(X)$.*

PROOF. Let $x \in X$. Then

$$\varphi^{-1}(\nu)(0) = \nu(\varphi(0)) \geq \nu(\varphi(x)) = \varphi^{-1}(\nu)(x).$$

Let $x_1, x_2, x_3 \in X$. As

$$\varphi^{-1}(\nu)(x_1 * x_3) = \nu(\varphi(x_1 * x_3)) = \nu(\varphi(x_1) * \varphi(x_3))$$
$$\geq T(\nu((\varphi(x_1) * \varphi(x_2)) * \varphi(x_3)), \nu(\varphi(x_2) * \varphi(x_3)))$$
$$= T(\nu(\varphi(x_1 * x_2) * x_3), \nu(\varphi(x_2 * x_3))) = T(\varphi^{-1}(\nu)((x_1 * x_2) * x_3), \varphi^{-1}(\nu)(x_2 * x_3))$$

so

$$\varphi^{-1}(\nu)(x_1 * x_3) \geq T(\varphi^{-1}(\nu)((x_1 * x_2) * x_3), \varphi^{-1}(\nu)(x_2 * x_3)).$$

Therefore $\varphi^{-1}(\nu) \in FPIIT(X)$. □

Acknowledgment

We would like to thank the referees for carefully reading the manuscript and making several helpful comments to increase the quality of the paper.

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Received : *October* 2021
Accepted : *December* 2021