Fuzzy Optimization and Modelling 3(1) (2022) 64-71

E-ISNN: 2676-7007

Contents lists available at FOMJ

Fuzzy Optimization and Modelling

Journal homepage: http://fomj.qaemiau.ac.ir/

Paper Type: Research Paper

Existence and Multiplicity Results for the $p(x)$ –Laplacian Equation via Genus Theory

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A R T I C L E I N F O

Article history:
Received 25 January 2022
Revised 09 March 2022
Accepted 11 March 2022
Available online 11 March 2022

Keywords:
$p(x)$ –Laplacian
Variational Method
Genus Theory
Sobolev Space

A B S T R A C T

In this paper, we study the existence and multiplicity of nontrivial weak solutions for the following equation involving weight and variable exponents

$$\begin{cases} -div\,(1+|\nabla u|^2)^{\frac{p(x)-2}{2}}\nabla u = \lambda m(x)|u|^{p(x)-2}u, & in\ \Omega, \\ u=0, & on\ \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth enough boundary which is subject to Dirichlet boundary condition, λ is a positive real parameter and p is real continuous function on $\overline{\Omega}$ with $1<p(x)<p^*(x)$, where $p^*(x)=\frac{Np(x)}{N-p(x)}$ and $p(x)<N$ for all $x\in\overline{\Omega}$, $m:\overline{\Omega}\to[0,\infty)$ is a continuous function.

By using a variational method and Krasnoselskii's genus theory, we show the existence and multiplicity of the solutions. For this purpose, we work on a generalized variable exponent Lebesgue-Sobolev space.

1. Introduction

In this paper, we study the following problem

$$\begin{cases} -div\,(1+|\nabla u|^2)^{\frac{p(x)-2}{2}}\nabla u = \lambda m(x)|u|^{p(x)-2}u, & in\ \Omega, \\ u=0, & on\ \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth enough boundary. Let λ be a positive real parameter and p be real continuous function on $\overline{\Omega}$ with $1<p(x)<p^*(x)$, where $p^*(x)=\frac{Np(x)}{N-p(x)}$ and $p(x)<N$ for all $x\in\overline{\Omega}$, $m:\overline{\Omega}\to[0,\infty)$ is a continuous function.

In recent years, elliptic problems involving variable exponent have been studied in many papers. In [6], they studied the eigenvalues of the $p(x)$-Laplacian Dirichlet problem:

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DOI: 10.30495/FOMJ.2022.1950685.1057

$$\begin{cases} -div\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda |u|^{p(x)-2}u, & in\ \Omega, \\ u = 0, & on\ \partial\Omega. \end{cases} \quad (2)$$

In [6], the authors showed that Λ, the set of eigenvalues, is a nonempty infinite set such that $\sup \Lambda = +\infty$.

Recently, Fan and Deng [4] studied the existence and multiplicity of positive solutions for the Neumann boundary value problem involving the $p(x)$-Laplacian of the following form:

$$\begin{cases} -div\left(|\nabla u|^{p(x)-2}\nabla u\right) + \lambda |u|^{p(x)-2}u = f(x,u), & in\ \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \eta} = \varphi, & on\ \partial\Omega. \end{cases} \quad (3)$$

They, under appropriate assumptions on f, obtained that the problem has at least two positive solutions.

In [1], the authors studied the Kirchhoff type equation:

$$\begin{cases} -M\left(\frac{1}{p(x)}\int_\Omega |\nabla u|^{p(x)}\,dx\right)\Delta_{p(x)}u = f(x,u), & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \quad (4)$$

by using the Krasnoselskii's Genus theory. They showed the existence and multiplicity of the solutions of the problem (4).

Here, we study the existence and multiplicity of the solutions for equation (1), using variational method and Krasnoselskii's Genus theory.

2. Preliminaries

First, we recall some necessary definitions and propositions concerning the Lebesgue and Sobolev spaces.

Let Ω be a bounded domain of $\mathbb{R}^N$. Set

$$C_+(\overline{\Omega}) := \{p \in C(\overline{\Omega}); p(x) > 1, \forall\, x \in \overline{\Omega}\}.$$

For any continuous function $p: \Omega \to (1,\infty)$,

$$p^- := \inf_{x\in\Omega} p(x) \qquad \text{and} \qquad p^+ := sup_{x\in\Omega}\, p(x).$$

For $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined by

$$L^{p(x)}(\Omega) := \left\{u: \Omega \to \mathbb{R}\ is\ a\ measurable\ function: \int_\Omega |u(x)|^{p(x)}dx < +\infty\right\}.$$

Endowed with the norm:

$$|u|_{p(x)} := inf\left\{\mu > 0: \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\},$$

$L^{p(x)}(\Omega)$ would be a separable reflexive Banach space [2].

The modular of $L^{p(x)}(\Omega)$ is defined by

$$\rho_{p(x)}(u) := \int_\Omega |u(x)|^{p(x)}dx.$$

Proposition 1. [8] The space $\left(L^{p(x)}(\Omega), |u|_{p(x)}\right)$ is separable, uniformly convex, reflexive and its conjugate space is $\left(L^{q(x)}(\Omega), |u|_{q(x)}\right)$, where $q(x)$ is the conjugate function of $p(x)$, i.e.,

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \forall x \in \Omega.$$

For all $u \in L^{p(x)}(\Omega)$, $v \in L^{q(x)}(\Omega)$, the Hölder's type inequality

$$\left|\int_{\Omega} uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)} |v|_{q(x)} \leq 2|u|_{p(x)}|v|_{q(x)}$$

holds.

Proposition 2. [11] Suppose that $u_n, u \in L^{p(x)}(\Omega)$, then the following properties hold:

$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}$;

$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}$;

$|u|_{p(x)} > 1(respectively, = 1; < 1) \Leftrightarrow \rho_{p(x)}(u) > 1(respectively, = 1; < 1)$;

$|u_n|_{p(x)} \to 0(respectively, \to +\infty) \Leftrightarrow \rho_{p(x)}(u_n) \to 0(respectively, \to +\infty)$;

$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0$.

The Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) := \{u \in L^{p(x)}(\Omega): |\nabla u| \in L^{p(x)}(\Omega)\},$$

is a separable and reflexive Banach spaces. For more details, we refer to [3].

Together the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

On $W^{1,p(x)}(\Omega)$, we may consider the following equivalent norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

$W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \inf\left\{\mu > 0: \int_{\Omega} \left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

It is well known that

$$W_0^{1,p(x)}(\Omega) := \{u \in L^{p(x)}(\Omega); u|_{\partial\Omega} = 0, |\nabla u| \in L^{p(x)}(\Omega)\}.$$

For more details, we refer to [2].

Proposition 3. (Sobolev Embedding [5]) For $p, q \in C_+(\overline{\Omega})$ such that $1 < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous compact embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is continuous and compact. Therefore, there is a constant $c_0 > 0$ such that

$$\|u\|_{q(x)} \leq c_0\|u\|.$$

Proposition 4. (Poincare Inequality [12]) There is a constant $c > 0$ such that

$$|u|_{p(x)} \leq c\|\nabla u\|_{p(x)}, \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

Remark 5. From proposition 4, $\|\nabla u\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

Definition 6. Let E be a real Banach space.

Set $R := \{A \subset E - \{0\}; A\ is\ compact\ and\ symmetric\}$. Let $A \in R$ and we define the genus of A as follows:

$\gamma(A) := \inf\{m \geq 1;\ \exists f \in C(A, \mathbb{R}^m \setminus \{0\}); f\ is\ odd\}$.

And $\gamma(A) = \infty$ if does not exist such a map f. $\gamma(\emptyset) = 0$ by definition. For more details, we refer to [7].

Theorem 7. [7] Let $\Omega \subset \mathbb{R}^N$ be a symmetric and bounded subset and $\partial\Omega$ as it's boundary. Assume that $0 \in \Omega$, then $\gamma(\partial\Omega) = N$.

Corollary 8. [7] The genus of unit sphere S^{N-1} of the space $\mathbb{R}^N$ is N.

3. Main results

Before the proceed the results, we need some notions:

Definition 9. $u \in W_0^{1,p(x)}(\Omega)$ is called a weak solution for (1) if

$$\int_\Omega (1 + |\nabla u|^2)^{\frac{p(x)-2}{2}} \nabla u \nabla v\, dx = \lambda \int_\Omega m(x)|u|^{p(x)-2} uv\, dx$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

The energy functional associated with problem (1) can obtained by

$$J(u) = \int_\Omega \frac{1}{p(x)} \left[(1 + |\nabla u|^2)^{\frac{p(x)}{2}} - (1 + |\nabla u|^2)^{\frac{p(x)-2}{2}} \right] dx - \lambda \int_\Omega \frac{m(x)}{p(x)} |u|^{p(x)}\, dx$$

for all $u \in W_0^{1,p(x)}(\Omega)$. It is well defined, C^1 functional and for all $u, v \in W_0^{1,p(x)}(\Omega)$

$$\langle J'(u), v \rangle = \int_\Omega (1 + |\nabla u|^2)^{\frac{p(x)-2}{2}} \nabla u \nabla v\, dx - \lambda \int_\Omega m(x)|u|^{p(x)-2} uv\, dx.$$

Therefore, critical points of this energy functional are week solutions for the problem (1).

We consider $\Omega \subset \mathbb{R}^N (N > 3)$ as a bounded domain with smooth boundary and $p \in C_+(\Omega)$ such that

$$1 < p^- \leq p(x) \leq p^+ < p^*(x), \tag{5}$$

and $p(x) < N$ for any $x \in \overline{\Omega}$.

(H) $m: \overline{\Omega} \to [0, \infty)$, $m \in L^\infty(\overline{\Omega})$.

Proposition 10. [8] The functional $\Lambda: W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by $\Lambda = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx$ is convex. The mapping $\Lambda': W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega)\right)^*$ is a strictly monotone, bounded homeomorphism and of (S_+) type, namely $u_n \rightharpoonup u$ (weakly) and $\overline{lim}_{n\to\infty}\left(\Lambda'(u_n), u_n - u\right) \leq 0$ implies $u_n \to u$ (strongly).

Definition 11. The functional J satisfies the Palais-Smale condition at the level $c \in \mathbb{R}$ ("$(PS)_c$ *condition*" for short) if for every sequence $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ satisfying

$$J(u_n) \to c \text{ and } J'(u_n) \to 0 \quad as\ n \to \infty,$$

has a convergence subsequence.

Theorem 12. [7] Let $J \in C^1(W_0^{1,p(x)}, \mathbb{R})$ and satisfies the $(PS)_c$ condition. We assume the following conditions:

i. J is bounded from below and even;

ii. There is a compact set $T \in R$ such that $\gamma(T) = k$ and $sup_{x\in T} J(x) < J(0)$.

Then problem (1) has at least k pairs of distinct critical points and their corresponding critical values are less than $J(0)$.

Theorem 13. If (5) and (H) hold. Then there are at least k pairs of distinct critical point (1).

(H_1) For $1 < p < +\infty$, we have $(a + b)^p \leq 2^{p-1}(a^p + b^p)$.

Lemma 14. We assume that (5) and (H) hold. Then J is coercive on $W_0^{1,p(x)}(\Omega)$ and bounded from below.

Proof. For any $u \in W_0^{1,p(x)}(\Omega)$, by (H[1]), we have

$$J(u) = \int_\Omega \frac{1}{p(x)}\left[(1+|\nabla u|^2)^{\frac{p(x)}{2}} - (1+|\nabla u|^2)^{\frac{p(x)-2}{2}}\right] dx - \lambda \int_\Omega \frac{m(x)}{p(x)}|u|^{p(x)}\,dx$$
$$\geq \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx - \frac{C}{p^-}\int_\Omega \left(1+|\nabla u|^{p(x)-2}\right) dx - \lambda\frac{\|m\|_\infty}{p^-}\int_\Omega |u|^{p(x)}\,dx$$
$$= \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx - \frac{C}{p^-}\int_\Omega dx - \frac{C}{p^-}\int_\Omega |\nabla u|^{p(x)-2}\,dx - \lambda\frac{\|m\|_\infty}{p^-}\int_\Omega |u|^{p(x)}\,dx$$
$$= \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx - \frac{C}{p^-}\int_\Omega |\nabla u|^{p(x)-2}\,dx - \lambda\frac{\|m\|_\infty}{p^-}\int_\Omega |u|^{p(x)}\,dx - \frac{C|\Omega|}{p^-},$$

$C > 0$ and $|\Omega|$ denote the Lebesgue measure of Ω.

If $\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)}dx$, by Proposition 4, we have some cases:

i. If $\rho_{p(x)}(u) > 1$,

$$J(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \frac{C}{p^-}\|u\|^{p^+-2} - \lambda\frac{\|m\|_\infty}{p^-}\|u\|^{p^+} - K.$$

Since (5), so J is coercive and bounded from below.

ii. If $\rho_{p(x)}(u) < 1$,

$$J(u) \geq \frac{1}{p^+}\|u\|^{p^+} - \frac{1}{p^-}\|u\|^{p^--2} - \lambda\frac{\|m\|_\infty}{p^-}\|u\|^{p^-} - K.$$

Since (5), so J is coercive and bounded from below. □

Lemma 15. The functional J satisfies the $(PS)_c$ condition.

Proof. We proceed by two steps:

Step1. We prove that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Let $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ be a $(PS)_c$ sequence. By contradiction we assume that, passing eventually to a subsequence, $\|u_n\| \to +\infty$ as $n \to +\infty$. We choose θ, $0 < \theta < \frac{1}{p^+}$.By definition 11 and (H[1]), for large enough n ,

$$C + \|u_n\| \geq J(u_n) - \theta\langle J'(u_n), u_n\rangle$$
$$= \int_\Omega \frac{1}{p(x)}\left[(1+|\nabla u_n|^2)^{\frac{p(x)}{2}} - (1+|\nabla u_n|^2)^{\frac{p(x)-2}{2}}\right] dx - \lambda\int_\Omega \frac{m(x)}{p(x)}|u_n|^{p(x)}\,dx$$
$$-\theta\int_\Omega (1+|\nabla u_n|^2)^{\frac{p(x)-2}{2}}|\nabla u_n|^2\,dx + \lambda\theta\int_\Omega m(x)|u_n|^{p(x)}\,dx$$
$$\geq \frac{1}{p^+}\int_\Omega |\nabla u_n|^{p(x)}\,dx - \frac{C}{p^-}\int_\Omega \left(1+|\nabla u_n|^{p(x)-2}\right)dx - \lambda\frac{\|m\|_\infty}{p^-}\int_\Omega |u_n|^{p(x)}\,dx$$
$$-\theta\int_\Omega |\nabla u_n|^{p(x)}\,dx + \lambda\theta\|m\|_\infty\int_\Omega |u_n|^{p(x)}\,dx$$
$$\geq \left(\frac{1}{p^+}-\theta\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx - \lambda\|m\|_\infty\left(\frac{1}{p^-}-\theta\right)\int_\Omega |u_n|^{p(x)}\,dx$$
$$-\frac{C}{p^-}\int_\Omega |\nabla u_n|^{p(x)-2}dx - \frac{C}{p^-}\int_\Omega dx$$
$$= \left(\frac{1}{p^+}-\theta\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx - \lambda\|m\|_\infty\left(\frac{1}{p^-}-\theta\right)\int_\Omega |u_n|^{p(x)}\,dx$$
$$-\frac{C}{p^-}\int_\Omega |\nabla u_n|^{p(x)-2}dx - \frac{C|\Omega|}{p^-},$$

$C > 0$ and $|\Omega|$ denote the Lebesgue measure of Ω.

Thus, the last inequality together with Proposition 4 and Remark 5, imply that

$$C + \|u_n\| \geq \left(\frac{1}{p^+} - \theta\right)\|u_n\|^{p^-} - \lambda\|m\|_\infty\left(\frac{1}{p^-} - \theta\right)\|u_n\|^{p^+} - \frac{C}{p^-}\|u_n\|^{p^+-2} - C_0 .$$

Dividing the above inequality by $\|u_n\|^{p^+}$, taking into account (5) holds and passing to the limit as $n \to +\infty$, we obtion a contradiction. It follows that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

Step 2. We prove that $\{u_n\}$ has a convergent subsequence in $W_0^{1,p(x)}(\Omega)$. It follows from proposition 3 and reflexivity of $W_0^{1,p(x)}(\Omega)$, we may assume that

$$u_n \rightharpoonup u \;\; in \;\; W_0^{1,p(x)}(\Omega), \;\; u_n \to u \;\; in \;\; L^{s(x)}(\Omega), \;\; u_n(x) \to u(x), \;\; a.e. \;\; in \; \Omega, \tag{6}$$

where $1 \leq s(x) < p^*(x)$.

By Hölder inequality and (6), we have

$$\left|\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)dx\right| \leq \int_\Omega |u_n|^{p(x)-1}|u_n-u|dx$$
$$\leq \left||u_n|^{p(x)-1}\right|_{\frac{p(x)}{p(x)-1}}|u_n-u|_{p(x)} \to 0 \quad \text{as} \quad n \to +\infty.$$

Thus,

$$\lim_{n\to+\infty}\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)dx = 0. \tag{7}$$

From Definition 11,

$$\langle J'(u_n), u_n - u\rangle \to 0.$$

Thus,

$$\langle J'(u_n), u_n - u\rangle = \int_\Omega (1+|\nabla u_n|^2)^{\frac{p(x)-2}{2}}\nabla u_n(\nabla u_n - \nabla u)\,dx$$
$$-\lambda\int_\Omega m(x)|u_n|^{p(x)-2}u_n(u_n-u)\,dx \to 0.$$

We can deduce from (7) that

$$\int_\Omega (1+|\nabla u_n|^2)^{\frac{p(x)-2}{2}}\nabla u_n(\nabla u_n - \nabla u)\,dx \to 0. \tag{8}$$

So by (8) and [13, Proposition 3.1] the sequence $\{u_n\}$ converges strongly to u in $W_0^{1,p(x)}(\Omega)$. Therefore, J satisfies the $(PS)_c$ condition.

□

Proof of Theorem 13. We notice that $W_0^{1,p^+}(\Omega) \subset W_0^{1,p(x)}(\Omega)$. Let us consider $(e_n)_{n=1}^{\infty}$ a schauder basis for $W_0^{1,p^+}(\Omega)$ [9] and $X_k = span\{e_1, e_2, \dots, e_k\}$, the subspace of $W_0^{1,p^+}(\Omega)$ generated by $e_1, e_2, \dots, e_k$. Clearly X_k is subspace of $W_0^{1,p(x)}(\Omega)$. Notice that $X_k \subset L^{p(x)}(\Omega)$ because $X_k \subset W_0^{1,p^+}(\Omega) \subset L^{p(x)}(\Omega)$. Thus, the norms $\|u\|$ and $\|u\|_{p(x)}$ are equivalent on X_k, since X_k is a finite dimension space [9].

Let $u \in X_k$; $\|u\| < 1$, from (H)

$$J(u) = \int_\Omega \frac{1}{p(x)}\left[(1+|\nabla u|^2)^{\frac{p(x)}{2}} - (1+|\nabla u|^2)^{\frac{p(x)-2}{2}}\right]dx - \lambda\int_\Omega \frac{m(x)}{p(x)}|u|^{p(x)}\,dx$$

$$\leq \frac{1}{p^-}\int_\Omega \left[(1+|\nabla u|^2)^{\frac{p(x)}{2}} - (1+|\nabla u|^2)^{\frac{p(x)-2}{2}}\right] dx - \lambda \frac{\|m\|_\infty}{p^+}\int_\Omega |u|^{p(x)}\, dx.$$

We consider some cases:

a. If $|\nabla u| > 1$, by (H₁)

$$J(u) \leq \frac{1}{p^-}\int_\Omega 2^{\frac{p(x)-1}{2}}|\nabla u|^{p(x)}\, dx - \lambda \frac{\|m\|_\infty}{p^+}\int_\Omega |u|^{p(x)}\, dx$$
$$\leq \alpha_1\|u\|^{p^-} - \alpha_2\|u\|^{p^+} = \|u\|^{p^-}\left[\alpha_1 - \alpha_2\|u\|^{p^+-p^-}\right].$$

Consider $r_1 \in (0,1)$ small enough in which ${r_1}^{p^-} < 1$ and $\alpha_1 \leq \alpha_2 \frac{{r_1}^{p^+-p^-}}{2}$.

Considering $T = S_r^k = \{u \in X_k;\ \|u\| = r_1\}, J(u) \leq {r_1}^{p^-}\left[\alpha_1 - \alpha_2 {r_1}^{p^+-p^-}\right], \forall u \in T$. Then

$$sup_T J(u) \leq \alpha_2 \frac{{r_1}^{p^+-p^-}}{2} - \alpha_2 {r_1}^{p^+-p^-} = -\alpha_2 \frac{{r_1}^{p^+-p^-}}{2} < 0 = J(0).$$

b. If $|\nabla u| < 1$

$$J(u) \leq \frac{1}{p^-}\int_\Omega 2^{\frac{p(x)}{2}}\, dx - \lambda \frac{\|m\|_\infty}{p^+}\int_\Omega |u|^{p(x)}\, dx \leq \alpha_1 - \alpha_2\|u\|^{p^+}.$$

Consider $r_1 \in (0,1)$ small enough in which ${r_1}^{p^+} < 1$ and $\alpha_1 \leq \alpha_2 \frac{{r_1}^{p^+}}{2}$.

Considering $T = S_r^k = \{u \in X_k;\ \|u\| = r_1\}, J(u) \leq \alpha_1 - \alpha_2 {r_1}^{p^+}, \forall u \in T$. Then

$$sup_T J(u) \leq \alpha_2 \frac{{r_1}^{p^+}}{2} - \alpha_2 {r_1}^{p^+} = -\alpha_2 \frac{{r_1}^{p^+}}{2} < 0 = J(0).$$

Since X_k and $\mathbb{R}^k$ are isomorphic so S_r^k and S^{k-1} are hemomorphic so $\gamma(S_r^k) = k$.

J is even, so by theorem 12, J has least k pairs of different critical points. □

Corollary 16. If (5) holds. Then there are infinitely many solution for (1).

Proof. Since k is arbitrary, so there are infinitely many critical points of J. □

4. Conclusion

In this paper we proved multiplicity existence of solutions for the problem (1), provided the parameter λ be positive, p a real continuous function on $\overline{\Omega}$ and $m: \overline{\Omega} \to [0,\infty)$ is a continuous function. In fact, using the Krasnoselskii's Genus theory we showed multiplicity solutions for problem (1).

Conflict of interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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https://doi.org/10.30495/FOMJ.2022.1950685.1057

Received: 25 January 2022 Revised: 09 March 2022 Accepted: 11 March 2022