International Journal of
Mathematical Modelling & Computations
Vol. 12, No. 03, Summer 2022, 163- 171
DOI:10.30495/ijm2c.2022.1950818.1243

Finding the Solution of a Nonlinear Matrix Problem by an Inverse-Free Iteration Scheme

T. Nadaf[a] and T. Lotfi[a,*]

[a] *Department of Mathematics, Hamedan Branch, Islamic Azad University, Hamedan, Iran.*

Abstract. In this work, an iterative method under the umbrella of inverse-free methods which do not rely on the calculation of the inverse matrix per loop is proposed for finding the maximal solution of a well-known nonlinear matrix equation (NME) in the form of Hermitian positive definite (HPD) matrices. The computational of the minimal solution is discussed as well. The iterative scheme is constructed based on methods for finding generalized matrix inverse. We illustrate some estimations for obtaining the solution, and its convergence. To ensure its validity and usefulness, some experiments are run which reveal the superiority of the proposed method.

Received: 26 January 2022, Revised: 01 March 2022, Accepted: 17 July 2022.

Keywords: Inverse-Free; Iteration Scheme; Nonlinear Matrix Equations; Hermitian Positive Definite.

AMS Subject Classification: 15A24, 65F30.

Index to information contained in this paper

1. Introduction

Here a nonlinear matrix equation (NME) [13] of the following type, where $\mathcal{P} \in \mathbb{C}^{n\times n}$ is its Hermitian Positive Definite (HPD) solution, is considered:

$$\mathcal{P} + A^*\mathcal{P}^{-1}A = I, \tag{1}$$

at which I is an identity matrix of suitable size, A is an $n \times n$ invertible complex or real matrix and A^* is the conjugate transpose of the matrix A, see [17]

*Corresponding author. Email: lotfi@iauh.ac.ir

The problem (1) is a part of the Riccati equation, often referred to as the Discrete-time Algebraic Riccati Equation or, for short, generally alluded to as DARE. In several disciplines such as stochastic filtering, queuing theory, etc., this type of nonlinear equation typically appears, see [1, 19] for further discussions.

This problem for any HPD solution $\mathcal{P}$ possesses the minimal solution $\mathcal{P}_S$ and the maximal solution $\mathcal{P}_L$ such that $\mathcal{P}_S \leqslant \mathcal{P} \leqslant \mathcal{P}_L$, see for more [7].

The maximal solution for (1) could be derived via:

$$\mathcal{P}_L = I - \mathcal{Y}_S,$$

where $\mathcal{Y}_S$ is the minimal solution of the dual NME as follows:

$$A\mathcal{Y}^{-1}A^* + \mathcal{Y} = I.$$

To discuss about the well-known NMEs given in literature, one may refer to Table 1.

Table 1. Various kinds of NMEs.

Year	NME
2001 [7]	$A^*\mathcal{P}^{-2}A + \mathcal{P} = \mathcal{I}$
2008 [9]	$\mathcal{P} = \mathcal{I} - A^*\mathcal{P}^{-1}A - \mathcal{B}^*\mathcal{P}^{-1}\mathcal{B}$
2009 [2]	$A^*\mathcal{P}^{-t}A + \mathcal{P}^s = \mathcal{Q}$
2013 [5]	$\mathcal{P} = \mathcal{I} - A^H\mathcal{P}^{-1}A - \mathcal{B}^H\mathcal{P}^{-1}\mathcal{B}$
2016 [6]	$\mathcal{P}_1 + A^*\mathcal{P}_2^{-\alpha_1}A = \mathcal{I}, \mathcal{P}_2 + \mathcal{B}^*\mathcal{P}_1^{-\alpha_2}\mathcal{B} = \mathcal{I}$
2017 [4]	$\mathcal{P} + A^*\mathcal{P}^{-1}A + \mathcal{B}^*\mathcal{P}^{-1}\mathcal{B} = \mathcal{Q}$

Solving the NMEs in Table 1 theoretically and finding the closed form solution is almost impossible and this opens up the avenue of finding numerical solutions by efficient iteration schemes which guarantee not only the convergence but also converge as fast as possible in low computational cost.

From now on we focus on (1) and review some of the existing methods for solving this NME. The fixed-point iterative method (FPI) is given by [3]:

$$\begin{cases} \mathcal{P}_0 = I, \\ \mathcal{P}_{k+1} = I - A^*\mathcal{P}_k^{-1}A. \end{cases}$$

Here per computing cycle, one matrix inverse must be computed. Authors in [10] presented another iteration scheme (SM) to compute the minimal solution as follows:

$$\begin{cases} \mathcal{P}_0 = AA^*, \\ \mathcal{P}_{k+1} = \mathcal{P}_k(2I - A^{-*}(I - \mathcal{P}_k)A^{-1}\mathcal{P}_k). \end{cases} \tag{2}$$

This scheme is categorized as methods without the computation of inverse per computing step, since A^{-1} is calculated once only. Noticing that $A^{-*} = A^{-1^*}$. Further discussions about nonlinear equation solvers can be found in [8, 16].

Iteration schemes such as the method (2) converges slowly specially at the beginning of the process due to the nature of local iteration schemes for solving nonlinear equations. In fact root solvers reaches the real convergence speed only when they arrive at the convergence phase which is not immediate if the initial guess/approximation is not chosen close enough to the simple roots. We have similar situations here when we tackle NMEs and employ them for solving NMEs. Author in [22] discussed how one can employ the Newton's methods for multiple roots in order to speed up the initial phase of the convergence for applying iterations schemes in solving NMEs as follows:

$$\mathcal{P}_{k+1} = \mathcal{P}_k \left((r+1)I - tH_k\mathcal{P}_k\right), \tag{3}$$

for any $1 \leqslant r \leqslant 2$.

An important iteration method presented in [14, 21], here denoted by EAM, is as follows:

$$\begin{cases} \mathcal{P}_0 = \mathcal{Z}_0 = I, \\ \mathcal{Z}_{k+1} = I + (I - \mathcal{P}_k)\mathcal{Z}_k, \\ \mathcal{P}_{k+1} = I - A^*\mathcal{Z}_{k+1}A. \end{cases}$$

Authors in [18] proposed another iteration scheme as a generalization of the Chebyshev's method (SOM) to solve (1) in what follows:

$$\begin{cases} \mathcal{P}_0 = AA^*, \\ \mathcal{P}_{k+1} = \mathcal{P}_k[3I - A^{-*}(I - \mathcal{P}_k)A^{-1}\mathcal{P}_k(3I - A^{-*}(I - \mathcal{P}_k)A^{-1}\mathcal{P}_k)]. \end{cases}$$

As can be seen the iterative methods discussed above are categorized in two forms. In one way, they need the calculation of inverse matrix per computing loop while for the other ones, they do not need. Since the calculation of an inverse matrix is computationally so expensive (see e.g. [11]), we attempt to propose a novel method which does not rely the computation of a matrix per cycle and needs only one matrix inverse at the beginning of the procedure. Some theoretical discussions about the convergence of the HPD solution for this NME under the new proposed iteration scheme are given. Computational works are done severely to show how the method converges rapidly to the solution as rapidly as possible by choosing a proper initial approximation for the iteration scheme.

The rest of this paper is unfolded as follows. In Section 2, the solution of this NME as the zero of an operator nonlinear equation is introduced. It is shown by way of illustration that the proposed solution method is useful and provides promising results. Several numerical experiments are investigated with implementation details in Section 3 to confirm its applicability and to compare with several schemes. Lastly, some concluding summaries are made in Section 4.

2. Deriving a new iteration scheme

It is recalled that an efficient procedure to compute the HPD solutions of (1) is to find the solution of the nonlinear equation, $\mathcal{U}(\mathcal{P}) = 0$, where [10]:

$$\mathcal{U}(\mathcal{P}) := -I + \mathcal{P} + A^*\mathcal{P}^{-1}A = 0. \tag{4}$$

The equation (4) could be re-written as follows [18]:

$$\begin{aligned}\mathcal{U}(\mathcal{P}) &= A^*\mathcal{P}^{-1}A + \mathcal{P} - I\\ &= -(I-\mathcal{P}) + A^*\mathcal{P}^{-1}A\\ &= -(A^{-*} - \mathcal{P}A^{-*})A^* + A^*\mathcal{P}^{-1}A\\ &= A^*\mathcal{P}^{-1}A - A(A^{-1}A^{-*} - A^{-1}\mathcal{P}A^{-*})A^*\\ &= (A^{-1}\mathcal{P}A^{-*})^{-1} - A(A^{-1}A^{-*} - A^{-1}\mathcal{P}A^{-*})A^*.\end{aligned}$$

It is clear that we can incorporate now a transformation as comes next:

$$L = A^{-1}\mathcal{P}A^{-*},$$

to further simplify the nonlinear operator equation:

$$\mathcal{G}(L) = L^{-1} - A(A^{-1}A^{-*} - L)A^* = 0.$$

For obtaining an iteration scheme to find the minimal HPD solutions, now it is necessary to solve the following:

$$L^{-1} - B = 0, \tag{5}$$

where

$$B = A(A^{-1}A^{-*} - L)A^*. \tag{6}$$

To contribute we employ (5)-(6) and propose the following method:

$$\begin{cases}\mathcal{P}_0 = AA^*,\\ H_k = A^{-*}(I-\mathcal{P}_k)A^{-1},\\ C_k = I - H_k\mathcal{P}_k,\\ \mathcal{P}_{k+1} = \mathcal{P}_k[I + C_k + C_k^2 + C_k^3 + C_k^4].\end{cases} \tag{7}$$

The method has actually been built based on the following fifth-order iteration scheme [15]:

$$\mathcal{Z}_{k+1} = \mathcal{Z}_k(I + \mathfrak{R}_k(I + \mathfrak{R}_k(I + \mathfrak{R}_k(I + \mathfrak{R}_k)))),$$

for computing generalized outer inverse of A, where $\mathfrak{R}_k = I - A\mathcal{Z}_k$.

The matrix iteration (7) requires one time to compute A^{-1} specially at the initial stage of the implementation and it provides the iteration scheme to be considered as inversion-free methods to solve (1), see also [12].

Designing new iterative methods without studying their convergence cannot be useful in practice. Due to this, in what follows we investigate the convergence of the proposed method

Theorem 2.1 Having the HPD matrix $\mathcal{P}_0 = AA^*$, the method (7) furnishes a sequence of Hermitian matrices.

Proof It is obvious that the seed AA^* is Hermitian, and $H_k = A^{-*}(I-\mathcal{P}_k)A^{-1}$. Therefore, $H_0 = A^{-*}A^{-1} - A^{-*}AA^*A^{-1}$, is also Hermitian, i.e., $H_0^* = H_0$. Now using inductive argument, we have

$$\begin{aligned}(\mathcal{P}_1)^* &= (\mathcal{P}_0[I + C_0 + C_0^2 + C_0^3 + C_0^4])^* \\ &= [\mathcal{P}_0 + \mathcal{P}_0(I - H_0\mathcal{P}_0) + \mathcal{P}_0(I - H_0\mathcal{P}_0)^2 \\ &\quad + \mathcal{P}_0(I - H_0\mathcal{P}_0)^3 + \mathcal{P}_0(I - H_0\mathcal{P}_0)^4]^* \\ &= \mathcal{P}_1.\end{aligned}$$

By considering $(\mathcal{P}_l)^* = \mathcal{P}_l$, $(l \geqslant k)$ we now show that

$$\begin{aligned}(\mathcal{P}_{l+1})^* &= (\mathcal{P}_l[I + C_l + C_l^2 + C_l^3 + C_l^4])^* \\ &= [\mathcal{P}_l + \mathcal{P}_l(I - H_l\mathcal{P}_l) + \mathcal{P}_l(I - H_l\mathcal{P}_l)^2 \\ &\quad + \mathcal{P}_l(I - H_l\mathcal{P}_l)^3 + \mathcal{P}_l(I - H_l\mathcal{P}_l)^4]^* \\ &= \mathcal{P}_{l+1}.\end{aligned} \tag{8}$$

Note that $H_l = (H_l)^*$ has been employed in (8). Now the conclusion is valid for any $l + 1$. Therefore, proof is ended. ■

Theorem 2.2 Assume that $\mathcal{P}_k$ and A are invertible matrices. Then the sequence $\{\mathcal{P}_k\}$ produced by (7) tends to the minimal solution of the NME (1) by having $\mathcal{P}_0 = AA^*$.

Proof We start by denoting that $H_k = A^{-*}(I - \mathcal{P}_k)A^{-1}$. Thus, we can write

$$\begin{aligned}I - H_k\mathcal{P}_{k+1} &= I - [A^{-*}(I - \mathcal{P}_k)A^{-1}](\mathcal{P}_k[I + C_k \\ &\quad + C_k^2 + C_k^3 + C_k^4]) \\ &= I - [A^{-*}(I - \mathcal{P}_k)A^{-1}](\mathcal{P}_k[I + (I - H_k\mathcal{P}_k) \\ &\quad + (I - H_k\mathcal{P}_k)^2 + (I - H_k\mathcal{P}_k)^3 + (I - H_k\mathcal{P}_k)^4]) \\ &= (I - H_k\mathcal{P}_k)^5.\end{aligned} \tag{9}$$

Now by using (9) and taking a norm from both sides, one attains that

$$\|I - H_k\mathcal{P}_{k+1}\| \leqslant \|I - H_k\mathcal{P}_k\|^5. \tag{10}$$

So, the inequality (10) can lead to convergence as long as the initial approximation read

$$\|I - H\mathcal{P}_0\| < 1.$$

This yields to the following condition

$$\|I - [A^{-*}(I - \mathcal{P})A^{-1}]\mathcal{P}_0\| < 1.$$

Now by employing the HPD initial approximation $\mathcal{P}_0 = AA^*$, one obtains that

$$\|A^{-*}\mathcal{P}A^*\| < 1,$$

which holds as long as $\mathcal{P}$ is the minimal HPD solution of (1). Also, due to

$$0 < \mathcal{P}_S = I - A^*\mathcal{P}_S^{-1}A = A^*[A^{-*}A^{-1} - \mathcal{P}_S^{-1}]A = A^*[\mathcal{P}_0^{-1} - \mathcal{P}_S^{-1}]A,$$

we obtain that $\mathcal{P}_0^{-1} > \mathcal{P}_S^{-1}$, thus $\mathcal{P}_0 < \mathcal{P}_S$. Hence, by mathematical induction, it is seen that $\{\mathcal{P}_k\}$ converges to $\mathcal{P}_S$. Now the proof ends. ■

The proposed method (7) tends to generalized outer inverse of a matrix with fifth order of convergence but this is not the order when tackling the NMEs. In fact it converges linearly to the solution but as will be discussed Section 3, it is much faster than the existing schemes in literature for finding maximal solutions.

To accelerate the convergence of the scheme specially at the initial phase of convergence, one may impose a scaling approach to force each iteration in order to arrive at the convergence phase much faster. Besides, a better initial matrix can also help to reduce the number of iterates to obtain the final solution.

3. Benchmarking

Computational efficiency and convergence behavior of PM (7) for the NME (1) are hereby illustrated on some numerical problems. All the compared methods here are written in the programming package Mathematica 12.0, [20]. Some comments are in order:

- All computations are performed in standard floating point arithmetic without applying any compilations. The matrix inversions whenever required are done using Mathematica 12.0 built-in command.
- We find the number of iterates to observe the convergence. In the codes we wrote to implement different schemes, we stop all the applied schemes when two successive iterations in the infinity norm is less than a tolerance as follows:

$$\|\mathcal{P}_{k+1} - \mathcal{P}_k\|_\infty < 10^{-4}.$$

- Here PM is employed along with (3) with $k = 2$ iterates and then (7) is used.

Example 3.1 This experiment put on test the effectiveness of PM for a 3×3 real matrix A defined as follows:

$$A = \begin{pmatrix} 0.2 & 0.14 & -0.01 \\ 0.1 & 0.12 & 0.1 \\ 0.14 & 0.02 & 0.4 \end{pmatrix},$$

to find the minimal solution of (1) which is given by

$$\mathcal{P}_S = \begin{pmatrix} 0.0663655 & 0.0427279 & 0.0405507 \\ 0.0427279 & 0.0448048 & 0.0815726 \\ 0.0405507 & 0.0815726 & 0.245427 \end{pmatrix}.$$

The results are given in Figure 1. We have used $r = 1.5$ for the PM and SOM.

Noting that PM converges to $\mathcal{P}_S$, therefore we employ other methods in computing the maximal solution of the dual problem $AX^{-1}A^* + \mathcal{P} - I = 0$ in our written implementations to have fair comparisons.

Computational results from Example 3.1 show that PM converges faster that the existing iterative methods for the same purpose. Our method requires fewer numbers of iterates to reach the stopping criterion. In fact under mild conditions, error analysis, convergence and stability of the scheme were observed based on the expected norm of the error.

$$\mathcal{P}_1 = \begin{pmatrix} 0.0669299 & 0.0420752 & 0.0375196 \\ 0.0420752 & 0.0428617 & 0.0759879 \\ 0.0375196 & 0.0759879 & 0.231133 \end{pmatrix},$$

$$\mathcal{P}_2 = \begin{pmatrix} 0.0658539 & 0.0424737 & 0.0404553 \\ 0.0424737 & 0.0445757 & 0.081174 \\ 0.0404553 & 0.081174 & 0.244185 \end{pmatrix},$$

$$\mathcal{P}_3 = \begin{pmatrix} 0.0663145 & 0.0426801 & 0.0404622 \\ 0.0426801 & 0.0447352 & 0.0814063 \\ 0.0404622 & 0.0814063 & 0.244995 \end{pmatrix},$$

$$\mathcal{P}_4 = \begin{pmatrix} 0.066357 & 0.042715 & 0.0405195 \\ 0.042715 & 0.0447822 & 0.081515 \\ 0.0405195 & 0.081515 & 0.245279 \end{pmatrix},$$

$$\mathcal{P}_5 = \begin{pmatrix} 0.0663637 & 0.0427246 & 0.0405424 \\ 0.0427246 & 0.0447987 & 0.0815569 \\ 0.0405424 & 0.0815569 & 0.245387 \end{pmatrix},$$

$$\mathcal{P}_6 = \begin{pmatrix} 0.0663655 & 0.0427279 & 0.0405507 \\ 0.0427279 & 0.0448048 & 0.0815726 \\ 0.0405507 & 0.0815726 & 0.245427 \end{pmatrix}.$$

Example 3.2 Here we compare different iteration schemes to find the minimal solution of (1) when A is a complex matrix as follows:

$$A = \frac{1}{2}\begin{pmatrix} 0.3 & -0.2 & 0.5 & 0.1 \\ 0.1 & -0.36 & i & 0.3 \\ 0.05 & -0.01 & -0.1 & 0.12 \\ 0.1 & i & 0.23i & 0.001 \end{pmatrix},$$

and the solution is:

$$\mathcal{P}_S = \begin{pmatrix} 0.104 + 9.64 \times 10^{-7}i & 0.049 - 0.120i & -0.002 - 0.002i & -0.013 + 0.043i \\ 0.049 + 0.120i & 0.364 + 0.00002i & 0.020 - 0.034i & 0.022 + 0.171i \\ -0.002 + 0.002i & 0.020 + 0.034i & 0.011 + 1.64 \times 10^{-6}i & -0.020 + 0.008i \\ -0.013 - 0.043i & 0.022 - 0.171i & -0.020 - 0.008i & 0.456 + 0.00006i \end{pmatrix}.$$

The numerical evidences are furnished in Figure 2. It too reveals that the proposed method is efficient in solving NME (1). We have used $r = 2$ for the PM and SOM.

Before ending this section, it is necessary to re-do Example 3.2 but this time with another $r = 3.2$. In fact, one time with $r = 3.2$, the multiple Newton's method is employed and then we switch to the PM and SOM for the rest of the iterates. The results for such a consideration is furnished in Figure 3. This confirms the usefulness of PM one more time for finding the maximal solutions.

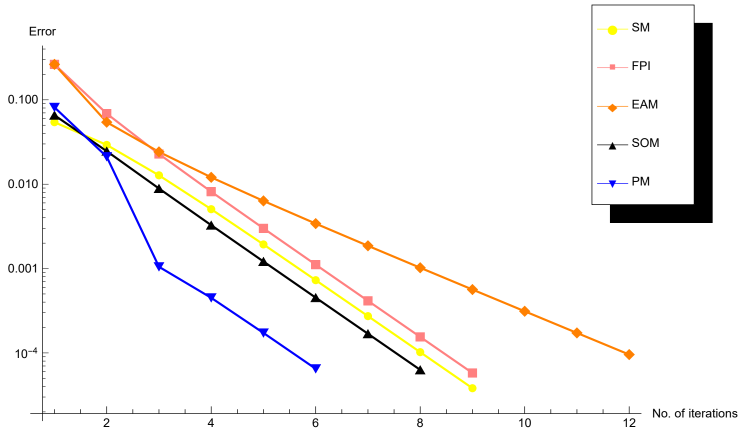

Figure 1. Comparison results in Example 3.1.

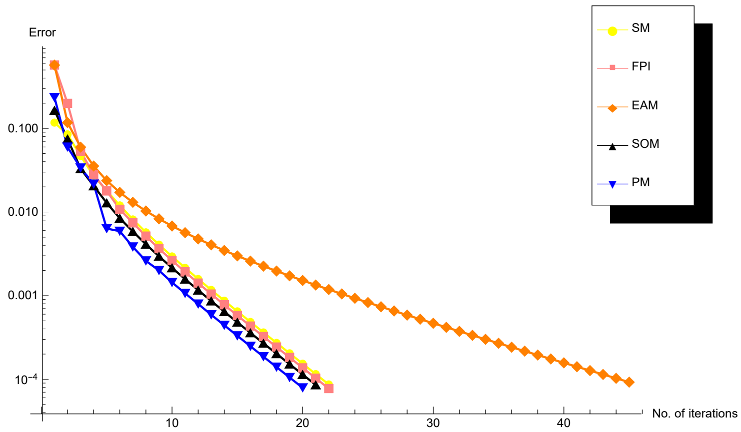

Figure 2. Comparison results in Example 3.2.

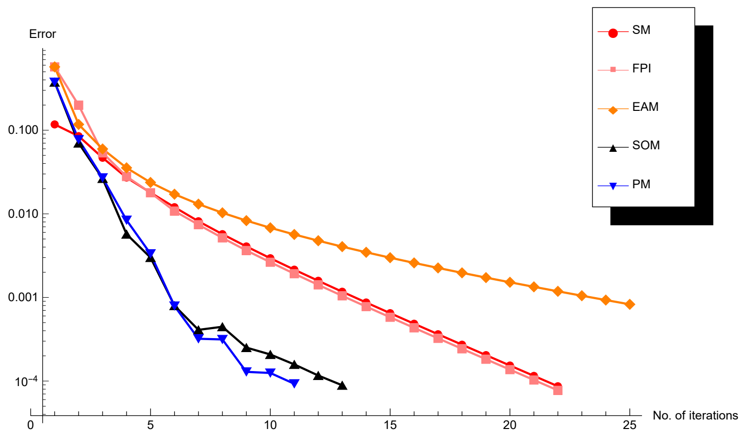

Figure 3. Comparison results in Example 3.2 using a different accelerator.

4. Conclusions

In this work, an iterative procedure for solving a NME was investigated and tested. Under mild conditions, error analysis and convergence of the scheme were established based on the expected norm of the error. To understand the scheme better, different types of numerical problems were given. All numerical tests show that the theoretical conclusions of this research are in agreement with the pieces of computational evidence. We drew the conclusion that the presented iterative technique is efficient in terms of computing computational solutions for a wide variety of nonlinear matrix problems. This technique also delivers accurate results with fewer iterations and cheaper computing costs when compared to several comparable iterative algorithms.

Acknowledgements

The authors are thankful to two anonymous referees for their comments and corrections on an earlier version of this manuscript.

References

[1] P. Bougerol, Kalman filtering with random coefficients and contractions, SIAM J. Control Optim., **31** (1993) 942–959.
[2] J. Cai and G. Chen, Some investigation on hermitian positive definite solutions the matrix equation $X^s + A^* X^{-t} A = Q$, Linear Alg. Appl., **430** (2009) 2448–2456.
[3] J.C. Engwerda, On the existence of a positive definite solution of the matrix equation $X + A^* X^{-1} A = I$, Linear Alg. Appl., **194** (1993) 91–108.
[4] V. I. Hasanov and A. A. Ali, On convergence of three iterative methods for solving of the matrix equation $X + A^* X^{-1} A + B^* X^{-1} B = Q$, Comput. App. Math., **36** (2017) 79–87.
[5] N. Huang and C.F. Ma, The inversion-free iterative methods for solving the nonlinear matrix equation $X = I - A^H X^{-1} A - B^H X^{-1} B$, Abst. Appl. Anal., **2013** (2013), Article ID 843785.
[6] N. Huang and C. Ma, J. Tang, The inversion-free iterative methods for a system of nonlinear matrix equations, Int. J. Comput. Math., **93** (2016) 1470–1483.
[7] I. G. Ivanov, V. I. Hasanov and B. V. Minchev, On matrix equations $X \pm A^* X^{-2} A = I$, Lin. Alg. Appl., **326** (2001) 27–44.
[8] M. Lalehchini, T. Lotfi and K. Mahdiani, Adaptive Steffensen-like methods with memory for solving nonlinear equations with the highest possible efficiency indices, Int. J. Indust. Math., **11** (2019) 337–345.
[9] J. H. Long, X.Y. Hu and L. Yhang, On the Hermitian positive definite solution of the nonlinear matrix equation $X = I - A^* X^{-1} A - \mathcal{B}^* X^{-1} B$, Bull. Braz. Math. Soc., **39** (2008) 371–386.
[10] M. Monsalve and M. Raydan, A new inversion-free method for a rational matrix equation, Linear Alg. Appl., **433** (2010) 64–71.
[11] K. Rezaei, F. Rahbarnia and F. Toutounian, A new approximate inverse preconditioner based on the Vaidya's maximum spanning tree for matrix equation $AXB = C$, Iran. J. Numer. Anal. Optim., **9** (2019) 1–16.
[12] A. Sadeghi, A Stable Iteration to the Matrix Inversion, Int. J. Math. Model. Comput., **08** (2018) 227–238.
[13] S.M. El-Sayed, An algorithm for computing positive definite solutions of the nonlinear matrix equation $X + A^* X^{-1} A = I$, Int. J. Comput. Math., **80** (2003) 1527–1534.
[14] S.M. El-Sayed and A.M. Al-Dbiban, A new inversion free iteration for solving the equation $X + A^* X^{-1} A = Q$, J. Comput. Appl. Math., **181** (2005) 148–156.
[15] S.K. Sen and S.S. Prabhu, Optimal iterative schemes for computing Moore-Penrose matrix inverse, Int. J. Syst. Sci., **8** (1976) 748–753.
[16] A.R. Soheili, F. Soleymani and M.D. Petković, On the computation of weighted Moore-Penrose inverse using a high-order matrix method, Comput. Math. Appl., **66** (2013) 2344–2351.
[17] F. Soleymani, M. Sharifi, S. Shateyi and F. Khaksar Haghani, An algorithm for computing geometric mean of two Hermitian positive definite matrices via matrix sign, Abst. Appl. Anal., **2014** (2014), Article ID: 978629.
[18] F. Soleymani, M. Sharifi, S. Karimi Vanani, F. Khaksar Haghani and A. Kiliçman, An inversion-free method for finding positive definite solution of a rational matrix equation, The Sci. World J., **2014** (2014), Article ID: 560931.
[19] Y. Tian and C. Xia, On the low-degree solution of the Sylvester matrix polynomial equation, J. Math., **2021** (2021), Article ID 4612177.
[20] M. Trott, The Mathematica Guide-Book for Numerics, Springer, New York, NY, USA, (2006).
[21] X. Zhan, Computing the extremal positive definite solutions of a matrix equation, SIAM J. Sci. Comput., **17** (1996) 1167–1174.
[22] L. Zhang, An improved inversion-free method for solving the matrix equation $X + A^* X^{-\alpha} A = Q$, J. Comput. Appl. Math., **253** (2013) 200–203.