# Trajectory Planning for Point-to-Point Motion using High-order Polynomials

Hossein Barghi Jond[1*], Vasif V. Nabiyev[2], Tuğrul Çavdar[2]
1- Young Researchers and Elite Club, Ahar Branch, Islamic Azad University, Ahar, Iran.
Email: h-barghi@iau-ahar.ac.ir (Corresponding author)
2- Department of Computer Engineering, Karadeniz Technical University, Trabzon, Turkey.
Email: vasif, ulduz@ktu.edu.tr

Received: February 2019 Revised: May 2019 Accepted: June 2019

**ABSTRACT:**
In this paper, a trajectory planning problem based on high-order polynomials is formulated for a point-to-point motion. The problem aims to find suitable polynomial trajectories that connect an initial to a final configuration while satisfying other specified constraints. The constraints are considered as zero velocity at the endpoint as well as a limitation on acceleration for the whole motion time. However, this problem is very difficult to trace more particularly when the number of coefficients (decision-making variables) is large. As a new approach, a high-order polynomial equation containing only two-term is proposed to generate suitable trajectories between two configurations. The advantage of the proposed polynomial is that it can be traced analytically in order to get solutions for the two independent coefficients in a closed-form. The motion simulations show that the resulting high-degree trajectories with two-term polynomial satisfy the mentioned constraints as well as they are continuous and smooth. Additionally, comparing outputs of Genetic Algorithm with the closed-form solutions for the problem show that closed-form expressions generate coefficients that are near optimal.

**KEYWORDS:** Trajectory Planning, Polynomial Trajectories, Velocity Constraint, Acceleration Constraint.

## 1. INTRODUCTION

When mobile robots need to traverse along a given path segment, there are usually infinite number of trajectories for a mobile robot to move from one point to another point. However, only a limited number of these trajectories are appropriate to be tracked while satisfying specified velocity and/or acceleration constraints. The appropriate trajectories are continuous, smooth, and possibly optimal with respect to some constraints.

Trajectory planning is an active research area for manipulator and mobile robotic applications and automation [1-5]. In [1], manipulator trajectory planning with two-term objective function, which contains one-term proportional to the integral of squared jerk as well as another term proportional to the total execution time, was studied. Although, the method in [1] is smooth enough, the trajectory is limited with fifth-degree B-splines. [2] and [3] studied trajectory planning problems for mobile manipulator. [2] has presented a search-based algorithm for generating time-optimal trajectories for a mobile manipulator. [3] solved the problem through an iterative method and generated a smooth trajectory under the torque and jerk constraints. [4] considered trajectories which were composed of linear (straight) and circular path segments for a spherical robot. The trajectories were formulated based on the differential equations of motion. [5] planned trajectories in the form of S-curve considering the robot velocity as a high-degree polynomial. The coefficients of the polynomial are determined by a curve interpolation method under the constraints of minimum time and limited jerk-acceleration.

Although, analytical approaches are preferable for analyzing trajectory planning problems when closed-form solution becomes complicated or impossible to discover, numerical methods can be used to get solutions. Environment-Gene evolutionary Immune Clonal Algorithm [6], Particle Swarm Optimization [7], Random Particles Optimization Algorithm [8] are some of the methods which are used to search numerical solution in such trajectory planning problems. Differential Evolution algorithm was also used for line and curve segment trajectories under velocity and acceleration constraints [9]. Genetic algorithm was used to calculate the coefficients for cubic polynomial trajectory fragments [10]. Another study trained a neural network with robot dynamics to obtain the approximate minimum time trajectories [11]. Analytical approaches for closed-form solutions were also focused, such as piecewise-constant polynomial trajectories were

obtained analytically in [12]. A more comprehensive review of the trajectory planning approaches and the techniques are presented in a survey [13].

This paper focuses on trajectory planning based on high-order polynomials. In general, using high-degree polynomial requires additional coefficients [14]. To reduce the calculation, this paper proposes a high-degree polynomial containing only two-term with any degree. In this condition, the problem will be tractable and the coefficients could be determined analytically in the closed-form expressions. The proposed approach aims to simplify the high-order polynomial by setting most of the coefficients as zero and to leave only two coefficients as decision variables. This simplification surely will reduce the computation, moreover will make the problem traced analytically. Closed-form solutions are studied in [15], [16]. Furthermore, even if this two-term simplification will limit the performance, it still provides the motivation of this paper because high-order polynomials are highly desirable due to their smoothed behavior. Additionally, the closed-form solutions by this approach give an important advantage because the motion controller of the vehicle can update trajectory commands in real time.

This study assumes that geometrical path-pieces are given either straight line or circular-arc. Therefore, trajectory problems for each straight and circular-arc path segments are formulated with the proposed two-term polynomial. The polynomial must satisfy initial and final velocity conditions as well as an acceleration constraint. Tracing the problem analytically and getting the polynomial coefficients in closed-form expressions have shown that this polynomial generates continuous and smooth trajectories.

The rest of this paper is arranged as follows. Section 2 describes how the trajectory planning problems based on the two-term polynomial are formulated and then, the analytical solution approaches are presented. The simulation results and the discussions are given in Section 3. Section 4 concludes the outcome of this research.

## 2. FORMULATION OF TRAJECTORY PLANNING PROBLEMS

Trajectory planning problems here are presented with the following assumptions. Firstly, a point mass model is considered for the mobile robot, therefore actual dynamic of the robot is neglected. Secondly, it is assumed that the mobile robot starts to move from origin of an Earth-fixed coordinate frame along a straight or a circular-arc path segment. Thirdly, the robot stops gently at the end of time or path. This condition is stated as zero velocity constraint at the endpoint/end time. Finally, acceleration/deceleration of the robot during motion is limited to safe navigation.

Accurate control of the robot along the path segments is highly dependent on the used motion model to design the robot controller. A common approach is to use the point mass model with acceleration-level control inputs. The motion controller designed based on this approach can be embedded in inner control loops at actuator-level that neglects the robot dynamics. In addition, the geometry of the path segment determines the acceleration constraint.

In accordance with the assumptions mentioned above, the trajectory planning problems using high-order polynomials for either straight line or circular-arc path segments can be formulated as follows. Consider a trajectory polynomial in form of

$$q(t) = \lambda_1 t^n + ... + \lambda_j t^2 \quad (1)$$

With respect to the endpoint velocity condition at final time $t_f > 0$,

$$\dot{q}(t_f) = 0 \quad (2)$$

As well as the acceleration constraint for $0 \le t \le t_f$,

$$|\ddot{q}(t)| \le \Phi \quad (3)$$

Where, $n \ge 3$, $\lambda_1 t^n$ is the leading term, $\lambda_1 \neq 0$ is the leading coefficient and $\Phi$ is a positive constant. $q(t)$ shows the covered distance along the path segment at time $t$. The problem is to find independent coefficients $\lambda_1, \ldots \lambda_j$ for (1) satisfying both (2) and (3).

The independent coefficients of (1) satisfying (2) and (3) cannot be often obtained (i.e. often the problem is not tractable) when $n$ and the number of terms are large. Therefore, to make the problem tractable and get the polynomial coefficients for any $n \ge 3$, this paper introduces the two-term trajectory polynomial in form of

$$q(t) = \lambda_1 t^n + \lambda_2 t^m \quad (4)$$

Where, $n - 1 \ge m \ge 2$.

In the next subsections, trajectory planning problems for straight and circular-arc path segments are rearranged according to (4) subject to (2) and (3), respectively. Then, the goal is to determine the two-term polynomial independent coefficients ($\lambda_1$ and $\lambda_2$) in order to obtain suitable trajectories for each path segment. Notice that the larger degrees of the terms in this polynomial cause to slower, but smoother trajectories while lower degrees of the terms give rise to faster but less smooth trajectories. The degrees can be chosen at the trade-off point of smoothness and motion speed according to the system requirements.

### 2.1. Straight Path Trajectories

Two trajectory planning problems are introduced here to find the polynomial coefficients $\lambda_1$ and $\lambda_2$.

*Problem 1*: consider two-term polynomial in (4) for motion along a straight path segment by assuming that the final time of the motion $t_f$ is known. Velocity endpoint condition in (2) for this problem can be rewritten as (5).

$$n\lambda_1 t_f^{n-1} + m\lambda_2 t_f^{m-1} = 0 \tag{5}$$

Moreover, acceleration constraint in (4) can be rearranged as,

$$\left| n(n-1)\lambda_1 t^{n-2} + m(m-1)\lambda_2 t^{m-2} \right| \le \Phi . \tag{6}$$

The analytical approach is given below to find the closed-form solutions for $\lambda_1$ and $\lambda_2$.

Firstly, $\lambda_1$ is left alone on the left side in the velocity boundary (5) as the following.

$$\lambda_1 = -\frac{m}{n}\lambda_2 t_f^{m-n} \tag{7}$$

Substituting it in the acceleration constraint (6), the inequation can be rearranged in terms of $\lambda_2$ as,

$$\left| m\lambda_2 t^{m-2}\left[ (m-1) - (n-1)\left(\frac{t}{t_f}\right)^{n-m} \right] \right| \le \Phi . \tag{8}$$

The left side of (8) is the acceleration function for time interval [0, $t_f$ ]. Here, it is necessary to find the critical points of the acceleration function in (8). Well-known *Fermat's theorem* states that "if $f(t)$ has a local extremum at a point $t = t^*$ and $f$ is differentiable at $t^*$, then derivative of $f$ at $t^*$ is zero." Using this theorem and because the acceleration function (left side of (8)) reaches to maximum or minimum at its critical points, the critical point of (8) could be obtained as follows.

$$t_{cp} = t_f \left[ \frac{(m-1)(m-2)}{(n-1)(n-2)} \right]^{\frac{1}{n-m}} \tag{9}$$

If (8) is satisfied at the time points 0, $t_{cp}$ and $t_f$, it is also satisfied during whole time interval [0, $t_f$ ]. Thus the variable $t$ in (8) is substituted with 0, $t_{cp}$ and $t_f$ , so that the nonlinearity of the inequation is eliminated. To satisfy (8), the inequation is rewritten for $t_{cp}$ and $t_f$ as (10) and (11), respectively.

$$\left| \frac{(n-m)(m-1)m}{n-2}\lambda_2 t_f^{m-2}\left[ \frac{(m-1)(m-2)}{(n-1)(n-2)} \right]^{\frac{m-2}{n-m}} \right| \le \Phi , \tag{10}$$

$$\left| (m-n)m\lambda_2 t_f^{m-2} \right| \le \Phi \tag{11}$$

(10) and (11) have only one variable which is $\lambda_2$. When $\lambda_2$ is left alone on the left side, (12) and (13) are obtained here in below.

$$\lambda_2 \le \frac{(n-2)\Phi}{(n-m)(m-1)mt_f^{m-2}}\left[ \frac{(n-1)(n-2)}{(m-1)(m-2)} \right]^{\frac{m-2}{n-m}}, \tag{12}$$

$$\lambda_2 \le \frac{\Phi}{(m-n)mt_f^{m-2}} \tag{13}$$

Here, choosing the minimum boundary value for $\lambda_2$ ensures satisfying (8) as follows.

$$\lambda_2 = \frac{\Phi}{mt_f^{m-2}} \min\left\{ \frac{(n-2)}{(n-m)(m-1)}\left[ \frac{(n-1)(n-2)}{(m-1)(m-2)} \right]^{\frac{m-2}{n-m}}, \frac{1}{m-n} \right\} \tag{14}$$

(14) is the closed-form solution for $\lambda_2$. As $\lambda_1$ is in hand from (7) in terms of $\lambda_2$, the trajectory polynomial (4) can be rearranged for $\lambda_2$ as

$$q(t) = -\frac{m}{n}t_f^{m-n}\lambda_2 t^n + \lambda_2 t^m \tag{15}$$

Where, $\lambda_2$ can be substitute from (14). Equation (15) generates suitable two-term polynomial trajectories for problem 1.

*Problem 2*: Consider a situation that a given distance of $Q$ has to be covered by (4) at a minimum time of $t_{f\min}$, with a given $Q$, (4) could be rearranged as

$$Q = \lambda_1 t_{f\min}^n + \lambda_2 t_{f\min}^n \tag{16}$$

In addition, velocity endpoint condition (3) can be rewritten as,

$$n\lambda_1 t_{f\min}^{n-1} + m\lambda_2 t_{f\min}^{m-1} = 0 . \tag{17}$$

For this problem, the analytical approach is given below to find the closed-form solutions for $\lambda_1$ and $\lambda_2$.

Firstly, getting $\lambda_1$ from velocity boundary (17) and substituting into (16), $t_{f\min}$ can be obtained as

$$t_{f\min}=\left[\frac{nQ}{(n-m)\lambda_2}\right]^{\frac{1}{m}}. \tag{18}$$

In addition, $t_{cp}$ could be obtained by using the same way in (9), as following

$$t_{cp}=t_{f\min}\left[\frac{(m-1)(m-2)}{(n-1)(n-2)}\right]^{\frac{1}{n-m}}. \tag{19}$$

When (18) is substituted into (19), $t_{cp}$ will be as,

$$t_{cp}=\left[\frac{nQ}{(n-m)\lambda_2}\right]^{\frac{1}{m}}\left[\frac{(m-1)(m-2)}{(n-1)(n-2)}\right]^{\frac{1}{n-m}}. \tag{20}$$

After the same equations from (10) to (13) of Problem 1 are obtained here, the closed-form solutions of $\lambda_2$ and $\lambda_1$ with substitution $t_{f\min}$ in (18) could be achieved as

$$\lambda_2=\frac{nQ}{n-m}\left[\frac{\Phi(n-m)}{Qnm}\min\left\{\frac{(n-2)}{(n-m)(m-1)}\left[\frac{(n-1)(n-2)}{(m-1)(m-2)}\right]^{\frac{m-2}{n-m}},\frac{1}{m-n}\right\}\right]^{\frac{m}{2}} \tag{21}$$

$$\lambda_1=-\frac{mQ}{n-m}\left[\frac{(n-m)}{Qn}\lambda_2\right]^{\frac{n}{m}} \tag{22}$$

Therefore, suitable two-term polynomial trajectories for this problem for the time interval [0, $t_f$] is given by

$$q(t)=-\frac{mQ}{n-m}\left[\frac{(n-m)}{Qn}\lambda_2\right]^{\frac{n}{m}}t^n+\lambda_2 t^m. \tag{23}$$

### 2.2. Circular-Arc Path Trajectories

In a circular path-piece, the total acceleration of the vehicle is composed of two vectors which are centripetal ($\vec{a}_c$) and tangential ($\vec{a}_t$). Therefore, the acceleration constraint in (3) must be written as in form of

$$\left|\vec{a}_c+\vec{a}_t\right|\le\Phi \tag{24}$$

Where |.| shows magnitude of a vector. (24) can be rearranged as (25) in terms of vehicle velocity $v$ and radius $c$ of the circle that the arc path belongs to it.

$$\sqrt{\left(\frac{v^2}{c}\right)^2+\left(\frac{dv}{dt}\right)^2}\le\Phi \tag{25}$$

Now similar to previous section, two trajectory planning problems can be defined to find suitable polynomials for circular-arc path.

*Problem 3*: consider the following

$$r(t)=c\left(\lambda_1 t^n+\lambda_2 t^m\right) \tag{26}$$

Two-term polynomial for motion along a circular-arc path segment in a given $t_f$. The velocity condition is same as (5) while the acceleration constraint is

$$\sqrt{\frac{\dot{r}^4(t)}{c^2}+\ddot{r}^2(t)}\le\Phi. \tag{27}$$

(27) can be explained as following,

$$\left[n\lambda_1 t^{n-1}+m\lambda_2 t^{m-1}\right]^4+\left[n(n-1)\lambda_1 t^{n-2}+m(m-1)\lambda_2 t^{m-2}\right]^2\le\frac{\Phi^2}{c^2}. \tag{28}$$

Similar to Problem 1, $\lambda_1$ is obtained from velocity endpoint condition (5) and substituted into (28) as follows,

$$\left[m\lambda_2 t^{m-1}\left(1-\left(\frac{t}{t_f}\right)^{n-m}\right)\right]^4+\left[m\lambda_2 t^{m-2}\left((m-1)-(n-1)\left(\frac{t}{t_f}\right)^{n-m}\right)\right]^2\le\frac{\Phi^2}{c^2} \tag{29}$$

Thus, the acceleration constraint becomes dependent on $\lambda_2$ and $t$ only. Afterwards, based on *Fermat's theorem*, the critical points $t_{cp}$ must be found from the acceleration function (that is left side of (29)). As it could be seen in the acceleration equation; since the degrees are even, it consists of two positive components. Therefore, acceleration is stated as increasing function and absolute extremum point does not exist. To satisfy the acceleration constraint, local extremum points need to be formed in the acceleration function. Otherwise, the problem cannot be traced analytically. Thus, the local extremum points are obtained by equating both components separately to zero to search the problem 3 solution. This point for the first component was found as $t_{cp}=t_f$, while it is found as term in (30) for second component.

$$t_{cp} = t_f \left( \frac{m-1}{n-1} \right)^{\frac{1}{n-m}} \tag{30}$$

To satisfy (29), the inequation is rewritten for $t_{cp}$ and $t_f$ as (31) and (32), respectively.

$$\left[ \left( \frac{m-1}{n-1} \right)^{\frac{m-1}{n-m}} \frac{n-m}{n-1} m t_f^{m-1} \lambda_2 \right]^4 \le \frac{\Phi^2}{c^2}, \tag{31}$$

$$\left[ m(m-n) t_f^{m-2} \lambda_2 \right]^2 \le \frac{\Phi^2}{c^2} \tag{32}$$

Subsequently, by leaving $\lambda_2$ alone on the left side of (31) and (32), it can be obtained,

$$\lambda_2 \le \frac{\Phi^{\frac{1}{2}} c^{-\frac{1}{2}}}{m t_f^{m-1}} \left( \frac{n-1}{m-1} \right)^{\frac{m-1}{n-m}} \frac{n-1}{n-m}, \tag{33}$$

$$\lambda_2 \le \frac{\Phi c^{-1}}{(m-n) m t_f^{m-2}}. \tag{34}$$

Here, choosing the minimum boundary value for $\lambda_2$ ensures satisfying (29) as follows

$$\lambda_2 = \min \left\{ \frac{\Phi^{\frac{1}{2}} c^{-\frac{1}{2}}}{m t_f^{m-1}} \left( \frac{n-1}{m-1} \right)^{\frac{m-1}{n-m}} \frac{n-1}{n-m}, \frac{\Phi c^{-1}}{(m-n) m t_f^{m-2}} \right\} \tag{35}$$

As the closed-form solutions of $\lambda_2$ and $\lambda_1$ are obtained from (35) and (7) respectively, then suitable trajectories for this problem are generated by

$$r(t) = c \left( -\frac{m}{n} t_f^{m-n} \lambda_2 t^n + \lambda_2 t^m \right). \tag{36}$$

*Problem 4*: consider a situation that a given arc length of $R$ has to be covered at minimum time of $t_{f\,min}$. With a given $R$, (26) could be rearranged as

$$R = c \left( \lambda_1 t_{f\,\min}^n + \lambda_2 t_{f\,\min}^n \right). \tag{37}$$

Substituting $\lambda_1$ from velocity condition (5) into the arc length formula (37), $t_{f\,min}$ can be obtained as

$$t_{f\,\min} = \left[ \frac{nR}{c(n-m)\lambda_2} \right]^{\frac{1}{m}} \tag{38}$$

Moreover, $t_{cp}$ is obtained same in (30). When substituted (38) into $t_{cp}$ formula (in (30)), it can be obtained,

$$t_{cp} = \left[ \frac{nR}{c(n-m)\lambda_2} \right]^{\frac{1}{m}} \left( \frac{m-1}{n-1} \right)^{\frac{1}{n-m}}. \tag{39}$$

The same equations from (31) to (34) of the Problem 3 are obtained here, the closed-form solutions of $\lambda_2$ and $\lambda_1$ with substitution $t_{f\,min}$ in (38) are

$$\lambda_2 = \frac{nR}{c(n-m)} \min \left\{ \left[ \frac{\Phi^{\frac{1}{2}} c^{\frac{1}{2}}}{R} \left( \frac{n-1}{m-1} \right)^{\frac{m-1}{n-m}} \frac{n-1}{nm} \right]^m, \left( \frac{\Phi}{mnR} \right)^{\frac{m}{2}} \right\}, \tag{40}$$

$$\lambda_1 = -\frac{mR}{c(n-m)} \left( \frac{(n-m)c}{nR} \lambda_2 \right)^{\frac{n}{m}}. \tag{41}$$

Therefore, suitable trajectories for this problem for [0, $t_f$] can be generated by using (40) and (41).

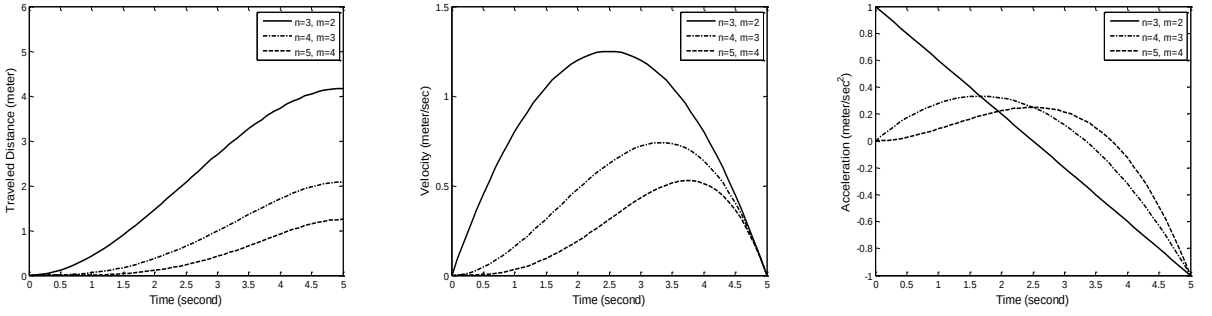
## 3. SIMULATION RESULTS AND DISCUSSIONS

While running on the obtained polynomial trajectories from previous section, it is necessary to investigate whether the generated trajectories satisfy the velocity conditions as well as the acceleration constraint during the whole time interval [0, $t_f$]. Consequently, several experiments are prepared to observer the behavior of the obtained two-term polynomials. In the experiment motion simulations, parameters of problems are defined as $t_f = 5$ seconds, $\Phi = 1$ m/s$^2$, $c = 1$ meter and $Q = R = 5$ meters. The profiles of traveled distance, velocity and acceleration in the 3rd, 4thand 5th degree of the two-term polynomials are shown in Fig. 1 to 4. For Problems 1 and 3, plots of the trajectories related with the degrees $n$: 3, 4, 5 are shown in Fig. 1 and 2, respectively. The travelled distance decreases as the degrees of the polynomials increase in Fig. 1a and 2a. The velocity boundary conditions are satisfied in the beginning and at the end of motion as seen in Fig. 1b and 2b. The acceleration constraint is satisfied during the whole time interval [0, $t_f$] as shown in Fig. 1c and 2c. For Problems 2 and 4, plots of the trajectories with the degrees $n$: 3, 4, 5 are shown in Fig. 3 and 4, respectively. The travelling time increases as the degrees of the polynomials increase in Fig. 3a and 4a. The velocity boundary conditions and the acceleration constraint are also satisfied as seen in Fig. 3b, 4b and Fig. 3c, 4c, respectively.

Furthermore, all four problems are also solved through Genetic Algorithm Toolbox of MATLAB platform. When both results are compared in Tables 1 to 4, it is explicit that the closed-form solutions are near to GA results. As seen in Tables 1 and 2, GA becomes

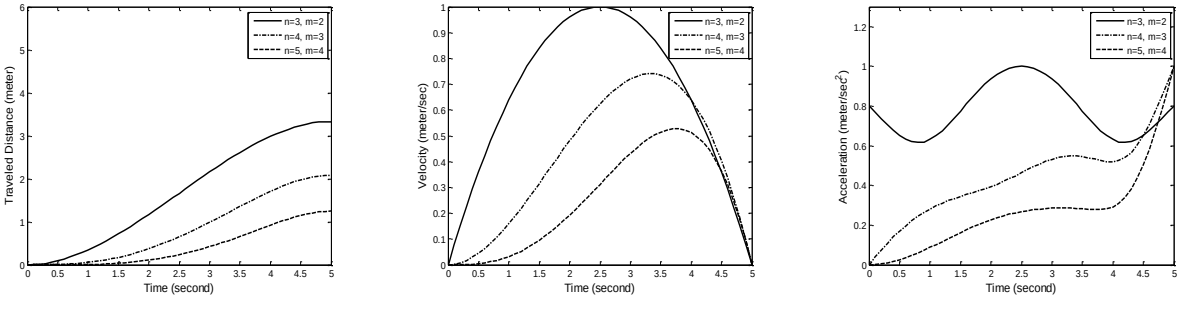
unable to reach a solution in the Problems 2 and 4 in the case of the difficult non-linear acceleration constraints.

Tables 3 and 4, show that the travelled distance decreases as the degrees of the polynomials increase in the Problems 1 and 3. Tables 2 and 4, show that the travelling time increases as the degrees of the polynomials increase in the Problems 2 and 4.



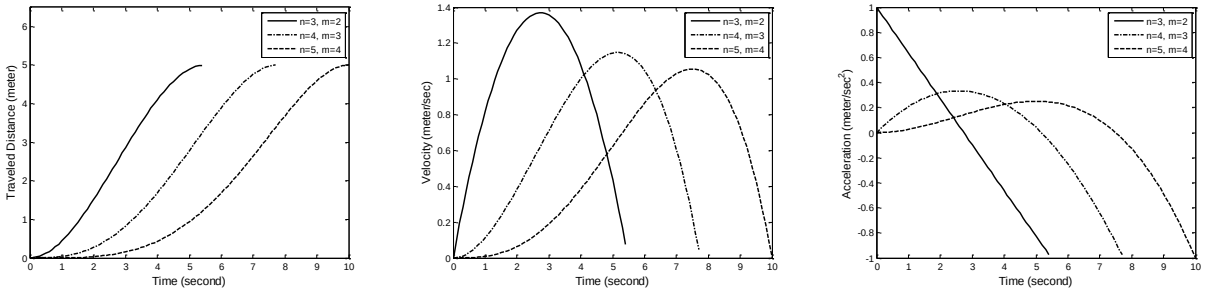

a b c

**Fig. 1.** Plots for the trajectories obtained in Problem 1, a. travelled distances, b. velocities, c. accelerations.



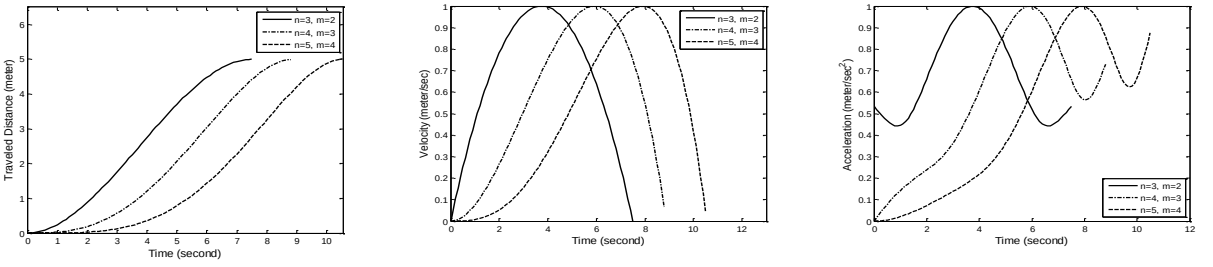

a b c

**Fig. 2.** Plots for the trajectories obtained in Problem 3, a. travelled distances, b. velocities, c. accelerations.




a b c

**Fig. 3.** Plots for the trajectories obtained in Problem 2, a. travelled distances, b. velocities, c. accelerations.




a b c

**Fig. 4.** Plots for the trajectories obtained in Problem 4, a. travelled distances, b. velocities, c. accelerations.

**Table 1.** Closed-form and GA results for Problem 2.

| $n$ | $m$ | Closed-form Solutions | | | GA Solutions | | |
|---|---|---|---|---|---|---|---|
| | | $\lambda_2$ | $\lambda_1$ | $t_{f\,min}$ | $\lambda_2$ | $\lambda_1$ | $t_{f\,min}$ |
| 3 | 2 | 0.500000 | -0.060858 | 5.477225 | 0.468100 | -0.055100 | 5.661000 |
| 4 | 3 | 0.043033 | -0.004166 | 7.745966 | 0.043200 | -0.004200 | 7.735200 |
| 5 | 4 | 0.002500 | -0.000200 | 10.00000 | No feasible solution found | | |

**Table 2.** Closed-form and GA results for Problem 4.

| $n$ | $m$ | Closed-form Solutions | | | GA Solutions | | |
|---|---|---|---|---|---|---|---|
| | | $\lambda_2$ | $\lambda_1$ | $t_{f\,min}$ | $\lambda_2$ | $\lambda_1$ | $t_{f\,min}$ |
| 3 | 2 | 0.500000 | -0.060858 | 5.477225 | 0.500000 | -0.060900 | 5.477200 |
| 4 | 3 | 0.043033 | -0.004166 | 7.745975 | 0.043000 | -0.004200 | 7.746000 |
| 5 | 4 | 0.002500 | -0.000200 | 10.00000 | 0.025000 | -0.000200 | 10.00000 |

**Table 3.** Closed-form and GA results for Problem 1.

| $n$ | $m$ | Closed-form Solutions | | | GA Solutions | | |
|---|---|---|---|---|---|---|---|
| | | $\lambda_2$ | $\lambda_1$ | $q(t_f)$ | $\lambda_2$ | $\lambda_1$ | $q(t_f)$ |
| 3 | 2 | 0.500000 | -0.066666 | 4.166666 | 0.500500 | -0.066700 | 4.179200 |
| 4 | 3 | 0.066666 | -0.010000 | 2.083333 | 0.067700 | -0.010100 | 2.147900 |
| 5 | 4 | 0.010000 | -0.001600 | 1.250000 | 0.010000 | -0.001600 | 1.376000 |

**Table 4.** Closed-form and GA results for Problem 3.

| $n$ | $m$ | Closed-form Solutions | | | GA Solutions | | |
|---|---|---|---|---|---|---|---|
| | | $\lambda_2$ | $\lambda_1$ | $r(t_f)$ | $\lambda_2$ | $\lambda_1$ | $r(t_f)$ |
| 3 | 2 | 0.400000 | -0.053333 | 3.333333 | 0.379300 | -0.050500 | 3.169100 |
| 4 | 3 | 0.066666 | -0.010000 | 2.083333 | 0.086600 | -0.012900 | 2.738600 |
| 5 | 4 | 0.010000 | -0.001600 | 1.250000 | 0.009200 | -0.001400 | 1.276100 |

## 4. CONCLUSIONS

In this paper a common polynomial trajectory planning problem considered subject to velocity and acceleration constraints. The common problem is simplified with taking a two-term polynomial instead. Then, the common problem is replaced with four sub problems, two for straight path segments and two for circular-arc path type. An analytical approach offering a closed-form solution is presented to find the coefficients of the two-term polynomial for all sub problems while taken into consideration velocity conditions as well as the acceleration constraint. By simulating the resulted two-term polynomial trajectories through the closed-form solutions, the accuracy of the analytical approach used is also ensured.

## REFERENCES


[1] Gasparetto and V. Zanotto, **"A New Method for Smooth Trajectory Planning of Robot Manipulators,"** *Mechanism and Machine Theory*, Vol. 42, pp. 455-471, 2007.

[2] S. Thakar, L. Fang, B. Shah and S. Gupta, **"Towards Time-Optimal Trajectory Planning for Pick-and-Transport Operation with a Mobile Manipulator, "** *IEEE 14th International Conference on Automation Science and Engineering (CASE)*, Munich, Germany, 2018.

[3] M. H. Korayem, V. Azimirad, B. Tabibian and M. Abdolhasani, **"Analysis and Experimental Study of Non-Holonomic Mobile Manipulator in Presence of Obstacles for Moving Boundary Condition,"** *Acta Astronautica*, Vol. 67, pp. 659-672, 2010.

[4] M. R. Azizi and D. Naderi, **"Dynamic Modeling and Trajectory Planning for a Mobile Spherical Robot with a 3Dof Inner Mechanism,"** *Mechanism and Machine Theory*, Vol. 64, pp. 251-261, 2013.

[5] E. Red, **"A Dynamic Optimal Trajectory Generator for Cartesian Path Following,"** *Robotica*, Vol. 18, pp. 451-458, 2000.

[6] H. Xu, J. Zhuang, S. Wang and Z. Zhu, **"Global Time-Energy Optimal Planning of Robot Trajectories,"** *IEEE International Conference on Mechatronics and Automation, Changchun*, China, 2009.

[7] P. Huang and Y. Xu, **"PSO-based Time-Optimal Trajectory Planning for Space Robot With Dynamic Constraints,"** *IEEE International Conference on Robotics and Biomimetics, Kunming, China*, 2006.

[8] Mohajer, K. Kiani, E. Samiei and M. Sharifi, **"A New Online Random Particles Optimization Algorithm for Mobile Robot Path Planning in Dynamic Environment,"** *Mathematical Problems in Engineering*, Vol. 2013, pp. 1-9, 2013.

[9] S. Aydin and H. Temeltas, **"Fuzzy-Differential Evolution Algorithm for Planning Time-Optimal Trajectories of a Unicycle Mobile Robot on a Predefined Path,"** *Advanced Robotics*, Vol. 18, pp. 725-748, 2004.

[10] F. J. Abu-Dakka, F. J. Valero, J. L. Suner and V. Mata, **"Direct Approach to Solving Trajectory Planning Problems using Genetic Algorithms with Dynamics Considerations In Complex Environments, "** R*obotica*, Vol. 33, pp. 669-683, 2015.

[11] G. Fang and M. W. M. G. Dissanayake, **"A Neural Network-Based Method for Time-Optimal Trajectory Planning,"** *Robotica*, Vol. 16, pp. 143-158, 1998.

[12] Z. Qu, J. Wang and C. E. Plaisted, **"A New Analytical Solution to Mobile Robot Trajectory Generation in the Presence of Moving Obstacles,"** *IEEE Transactions on Robotics*, Vol. 20, pp. 978-993, 2004.

[13] A. Gasparetto, P. Boscariol, A. Lanzutti and R. Vidoni, **"Trajectory Planning in robotics,"** *Mathematics In Computer Science*, Vol. 6, pp. 269–279, 2012.

[14] M. Boryga and A. Grabos, **"Planning of Manipulator Motion Trajectory with Higher-Degree Polynomials Use,"** *Mechanism and Machine Theory*, Vol. 44, pp. 1400–1419, 2009.

[15] H. B. Jond, V. V. Nabiyev and R. Benveniste, **"Trajectory Planning Using High Order Polynomials under Acceleration Constraint,"** *Journal of Optimization in Industrial Engineering*, Vol. 21, pp. 1-6, 2017.

[16] H. B. Jond, V. V. Nabiyev and A. Akbarimajd, **"Planning of Mobile Robots under Limited Velocity and Acceleration,"** *Signal Processing and Communications Applications Conference (SIU*), Trabzon, Turkey, 2014.