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Research Article

A New Approach to Solving Multi-follower Multi-objective Linear Bilevel Programming Problems

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Abstract

In this paper, we present a suitable extension of the approach described by Pieume et al. (2011) for solving multi-follower multi-objective linear bilevel programming problems. This problem is a special case of multi-follower bilevel linear programming problems, where each decision maker possesses several objective functions that in some cases, conflict with one another. We construct a multi-objective linear programming problem. Furthermore, we show that the multi-follower multi-objective linear bilevel programming problem can be reduced to optimize the top-level multi-objective linear programming problem over an efficient set. The proposed approach uses a Pareto-filter scheme, and obtains an approximate discrete representation efficient set unlike the fuzzy approaches that only obtain one efficient solution. Ultimately, a numerical example is presented to illustrate the efficiency of the proposed approach.

Keywords: Multi-objective programming; Multi-follower linear bilevel programming; Pareto-optimal solutions; Feasible set

1. Introduction

A bilevel programming (BLP) problem can be considered as a non-cooperative two-player game, which was first presented by Von Stackelberg (1952). In a basic BLP model, controlling the decision variables is partitioned among the players, who seek to optimize their individual objective functions. As two definitions, the upper level is called the leader and the lower level is termed as the follower. The game, which is cited as static, implies that each player has only one move. The leader acts first and attempts to optimize his objective function; then, the follower reacts in a way that is individually regardless of the extremural effects stemmed from observing the decision of leader. Many papers have been published investigating the results and solution methods for BLP problems (Bard, 1998; Dempe, 2003).

BLP problems occur in diverse applications, such as economics, transportation, engineering, and some other areas (Ma et al., 2014; Safaei et al., 2018; Me et al., 2013; Mehdizadeh & Mohamadizadeh, 2013; Mohagheghian et al., 2018). However, when one encounters a real world bilevel decision problem, the leader and the follower may have multiple conflict objectives that ought to be optimized simultaneously for achieving an efficient solution (Yin, 2000). BLP problems may involve multiple decision makers at the lower level, and these followers may have different reactions to a possible decision taken by the leader. This problem is called a bilevel multi-follower programming problem. Moreover, there exist different kinds of relationships amongst the followers (Zhang et al., 2016). Also, one can find other approaches in the literature dealing with multi-follower multi-objective bilevel problems (Ansari & Zhiani Rezai, 2011; Bkay, 2009; Farahi & Ansari, presented an algorithm for solving bilevel multi-objective decision-making with multiple interconnected decision makers. Zhang et al. (2008) developed an approximation branch-and-bound algorithm to solve a fuzzy multi-follower multi-objective bilevel problem. Zhang et al. (2010) solved the problem of fuzzy bilevel multi-follower multi-objective using the Kth-best method. Taran & Roghanian (2013) proposed a method for a fuzzy multi-objective multi-follower linear BLP problem to supply chain optimization. Habibpoor (2016) represented a new method for solving the linear bilevel multi-objective multi-follower programming problems. Lachhwani (2018) proposed an alternative method based on fuzzy goal programming approach to obtaining a solution to the multi-objective linear bi-level multi-follower programming problems.

Note that many of the presented approaches are to solve a fuzzy multi-follower multi-objective bilevel problem. Thus, we have been incentivized to provide an approach to solving the multi-follower multi-objective bilevel problems (MMLBPP) without utilizing the fuzzy method. In this paper, we have extended the proposed approach for MMLBPP by Pieume et al. (2011). We have also introduced an artificial multi-objective linear programming problem, based on which the resolution of the MMLBPP can be reduced to optimize the top-level multi-objective linear programming problem over an efficient set. The paper is organized as follows:

In the next section, we will present a few basic concepts of multiobjective optimization. In section 3, the formulation of an MMLBPP is presented. Section 4 shows a relation between the feasible set of the upper level decisions and the set of Pareto-optimal solutions to a particular multi-objective programming problem. Section 5, presents an algorithm for solving MMLBPP. Finally, the paper is concluded in section 6.

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2. Efficient Points in Multi-objective Programming

A multi-objective programming problem (MOPP) is generally formulated as follows:

$$min_x \quad h(x) = \left(h_1(x), h_2(x), \dots, h_Q(x)\right), \qquad (1)$$

$$s.t. \quad x \in U.$$

where $h: R^n \to R^Q$ is a vector valued objective function and $U \subseteq R^n$ is the set of constraints.

Due to the fact that for $Q \geq 2$, there is not a canonical order in R^Q, as there is on R, one must define how objective function vector $(h_1(x), h_2(x), \dots, h_Q(x))$ must be compared to different alternatives $x \in U$. Closed pointed convex cones are generally used for the derivation of partial orders in the decision space. Let K be an arbitrary cone such that $K \subseteq R^Q$; then, the binary relation with respect to the cone K (denoted $\leq_K$) is defined as:

$$a \leq_K b \text{ if and only if } b - a \in K.$$

Due to the fact that it is impossible to find a solution to optimize all the objective functions simultaneously, a weaker concept, which is the concept of non-dominated point is applied.

Definition 2.1. A point $y_0 \in h(U)$ is a non-dominated point with respect to the cone K if and only if there does not exist a point $y \in h(U)$, $y \neq y_0$, such that $y \leq_K y_0$. If y^* is a non-dominated point with respect to the cone K; then, $x^* \in U$ such that $y^* = h(x^*)$ is called Pareto-optimal (or efficient) solution with respect to the cone K.

The following definition of efficient points is the most cited one in the literature (Mattson et al. 2004; Messac & Mattson, 2002; Sayin, 2003).

Definition 2.2. A feasible point $x^* \in U$ is called Pareto-optimal if there does not exist $x \in U$ such that

$$(h_1(x), h_2(x), \dots, h_Q(x)) \leq (h_1(x^*), h_2(x^*), \dots, h_Q(x^*)),$$

and

$$\left(h_1(x), h_2(x), \dots, h_Q(x)\right) \neq \left(h_1(x^*), h_2(x^*), \dots, h_Q(x^*)\right).$$

If x^* is Pareto-optimal, then $h(x^*)$ is called non-dominated point.

Let us remark that Definition 2.2 is a particular case of Definition 2.1, where the used cone is $R_+^Q \setminus \{0_Q\}$. Thus, the Pareto-optimal points are the solutions, which cannot be improved in one objective function without deteriorating their performance in at least one of the other objective functions. Throughout the paper, the set of efficient points of a multi-objective optimization problem defined by vector valued the objective function h on a feasible set U, with respect to a cone K, will be denoted by $E(h, U, \leq_K)$ and the corresponding non-dominated set will be denoted by $N(h, U, \leq_K)$.

As a drawback, for a majority of MOPPs, it is not convenient to obtain an exact description of the efficient set, which typically includes a very large or infinite number of points. In general, solving MOPPs consists of finding a finite subset of the efficient set and presenting them for the evaluation of the decision maker (DM). A set W is a good representation of the efficient set $E(h, U, \leq_K)$ if the following three conditions are fulfilled: W is finite and contains a reasonable number of points; non-dominated points corresponding to W in a way that do not miss a large portion of $N(h, U, \leq_K)$ (coverage criterion); and these points do not include points that are very close to each other (uniformity criterion).

The approaches that could generate a representative subset of the efficient set, while solving linear multi-objective optimization problems, can be found in Mattson et al. (2004), Messac & Mattson (2002) and Sayin (2003).

3. Formulation of an MMLBPP

A multi-follower bilevel programming problem can be modelled as follows:

$$min_{x \in \mathrm{X}} \quad \mathrm{F}(x, \mathrm{y}_1, \dots, \mathrm{y}_\mathrm{k}), \qquad (2)$$

$$s.t. \begin{cases} G(x) \leq 0, \\ y_1 \text{ solves } \begin{cases} min_{y_1 \in Y^1} \ f^1(x, y_1), \\ s.t. \quad g_1(x, y_1) \leq 0, \end{cases} \\ y_2 \text{ solves } \begin{cases} min_{y_2 \in Y^2} \ f^2(x, y_2), \\ s.t. \quad g_2(x, y_2) \leq 0, \end{cases} \\ \vdots \\ y_\mathrm{k} \text{ solves } \begin{cases} min_{y_\mathrm{k} \in Y^\mathrm{k}} \ f^\mathrm{k}(x, y_\mathrm{k}), \\ s.t. \quad g_\mathrm{k}(x, y_\mathrm{k}) \leq 0. \end{cases} \end{cases}$$

where $x \in X \subset R^{n_1}$; $y_i \in Y^i \subset R^{n_2}$ for all $i = 1,2, \dots, k$; $F: X \times Y^1 \times \dots \times Y^k \to R$; $F^i: X \times Y^i \to R$ for all $i = 1,2, \dots, k$ are the outer problem (leader) objective function and the inner problem (follower) objective function, respectively; $G: X \to R$; $g_i: X \times Y^i \to R$ for all $i \in \{12, \dots, k\}$ are inequality constraints. The vector x are decision variables controlled by the leader.

Our focus will be on the linear formulation of an MMLBPP, given as follows:

$$min_{x \in \mathrm{R}_+^{\mathrm{n}_1}} \mathrm{F}(x, y_1, \dots, y_\mathrm{k}) =$$
$$\left(\mathrm{C}_1(x, y_1, \dots, y_\mathrm{k}), \mathrm{C}_2(x, y_1, \dots, y_\mathrm{k}), \dots, \mathrm{C}_{\mathrm{m}_1}(x, y_1, \dots, y_\mathrm{k})\right) \ (3)$$

$$
s.t.\begin{cases}
A_1x \le b_1, \\
y_1 \text{ solves } \begin{cases} min_{y_1 \in R_+^{n_2}} f^1(y_1) = \left(c_1^{(1)}y_1, c_2^{(1)}y_1, \dots, c_{m_2}^{(1)}y_1\right), \\ s.t. \quad A_2^{(1)}x + A_3^{(1)}y_1 \le b_2^{(1)}, \end{cases} \\
y_2 \text{ solves } \begin{cases} min_{y_2 \in R_+^{n_2}} f^2(y_2) = \left(c_1^{(2)}y_2, c_2^{(2)}y_2, \dots, c_{m_2}^{(2)}y_2\right), \\ s.t. \quad A_2^{(2)}x + A_3^{(2)}y_2 \le b_2^{(2)}, \end{cases} \\
\vdots \\
y_k \text{ solves } \begin{cases} min_{y_k \in R_+^{n_2}} f^k(y_k) = \left(c_1^{(k)}y_k, c_2^{(k)}y_k, \dots, c_{m_2}^{(k)}y_k\right), \\ s.t. \quad A_2^{(k)}x + A_3^{(k)}y_k \le b_2^{(k)}, \end{cases}
\end{cases}
$$

where $C_i,\ i = 1,2,\dots,m_1$ are $n_1 + kn_2$-dimensional constant row vectors; $c_i^{(j)},\ i = 1,2,\dots,m_2,\ j = 1,2,\dots,k,$, are n_2-dimensional constant row vectors; b_1 is a p-dimensional constant column vector and b_2^i for $i \in \{1,2,\dots,k\}$ is a q-dimensional constant column vector; A_1 is a $p \times n_1$ constant matrix; A_2^i for $i \in \{1,2,\dots,k\}$ is a $q \times n_1$ constant matrix and A_3^i for $i \in \{1,2,\dots,k\}$ a $q \times n_2$ constant matrix.

Let us denote by $R_i(x)$, the set of rational responses of the ith-follower $(i = 1,2,\dots,k)$, for each decision of the leader. It is defined as the Pareto-optimal points of the following problem:

$$min_{y_i \in R_+^{n_2}} f^{(i)}(y_i) = \left(c_1^{(i)}y_i, c_2^{(i)}y_i, \dots, c_{m_2}^{(i)}y_i\right), \qquad (4)$$

$$\text{s.t.} \quad A_3^{(i)}y_i \le b_2^{(i)} - A_2^{(i)}x,\ i = 1,2,\dots,k,$$

with this notation, one has the following formulation of MMLBPP (3):

$$min_{x \in X}\ F(x, y_1, \dots, y_k) = \left(C_1(x, y_1, \dots, y_k), C_2(x, y_1, \dots, y_k), \dots, C_{m_1}(x, y_1, \dots, y_k)\right), \qquad (5)$$

$$\text{s.t.} \quad A_1x \le b_1,$$

$$y_i \in R_i(x),\ i = 1,2,\dots,k.$$

Also, by using the following representation for the feasible space of MMLBPP:

$$\Omega = \{(x, y_1, \dots, y_k) \in R_+^{n_1} \times R_+^{n_2} \times \dots \times R_+^{n_2}:\ A_1x \le b_1,\ y_i \in R_i(x),\ \text{for all } i = 1,2,\dots,k\}$$

One obtains the following optimistic formulation of MMLBPP:

$$min_{x \in X}\ F(x, y_1, \dots, y_k) = \left(C_1(x, y_1, \dots, y_k), C_2(x, y_1, \dots, y_k), \dots, C_{m_1}(x, y_1, \dots, y_k)\right), \quad (6)$$

$$\text{s.t.} \quad (x, y_1, \dots, y_k) \in \Omega.$$

In the sequel, we present a theoretical result, which will be used later on to derive an algorithm for solving the MMLBPP. Throughout all the paper, Z represents a set defined as follows:

$$Z = \{(x, y_1, \dots, y_k) \in R_+^{n_1} \times R_+^{n_2} \times \dots \times R_+^{n_2}:\ A_1x \le b_1,\ A_2^{(i)}x + A_3^{(i)}y_i \le b_2^{(i)},\ \forall\ i = 1,2,\dots,k\}$$

It is assumed that Z is a non-empty and a bounded set over the convex polyhedron.

4. A New Characterization of the Feasible Set of an MMLBPP

Let us consider the following multi-objective linear programming problem:

$$min_{x, y_1, \dots, y_k}\ H(x, y_1, \dots, y_k) = \begin{pmatrix} 0 & D_1 & 0 & \cdots & 0 \\ 0 & 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D_k \\ -I & 0 & 0 & \cdots & 0 \\ e^t & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x \\ y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix}, \quad (7)$$

$$\text{s.t.} \quad (x, y_1, \dots, y_k) \in Z.$$

For $i = 1,2,\dots,k,\ D_i$ is a $m_2 \times n_2$ matrix with rows $c_j^{(i)}$, $j = 1,2,\dots,m_2$, which represents the objective functions row vector of the ith-follower; e is a vector having each entry equal to 1 and I is an $n_1 \times n_1$ identity matrix.

Let $K_2 = R_+^{m_2+n_1+1} \setminus \{0_{m_2+n_1+1}\}$, then the following result holds.

Theorem 4.1 $\Omega = E(H, Z, \le_{K_2})$.

Proof. $\Longleftarrow$ First we show that $E(H, Z, \le_{K_2}) \subset \Omega$.

Let $z = (x_0, y_1^{(0)}, y_2^{(0)}, \dots, y_k^{(0)}) \in E(H, Z, \le_{K_2})$ be arbitrary. From the definition of $E(H, Z, \le_{K_2})$, one has naturally $A_2^{(i)}x + A_3^{(i)}y_i \le b_2^{(i)},\ i = 1,2,\dots,k$ and $A_1x_0 \le b_1$. So, in order to show that $z \in \Omega$, it suffices to show that $y_i^{(0)} \in R_i(x_0)$, for all $i = 1,2,\dots,k$. Let us suppose the contrary, i.e., there exists $r \in \{1,2,\dots,k\}$ such that $y_r^{(0)} \notin R_r(x_0)$. Then, there exists $\bar{y}_r$ such that $A_2^{(r)}x_0 + A_3^{(r)}\bar{y}_r \le b_2^{(r)}$, and $\bar{y}_r$ dominates $y_r^{(0)}$, i.e.,

$$(c_1^{(r)}\bar{y}_r,\ c_2^{(r)}\bar{y}_r, \dots ,\ c_{m_2}^{(r)}\bar{y}_r) \le (c_1^{(r)}y_r^{(0)}, c_2^{(r)}y_r^{(0)}, \dots, c_{m_2}^{(r)}y_r^{(0)}),$$

and

$$(c_1^{(r)}\bar{y}_r,\ c_2^{(r)}\bar{y}_r, \dots ,\ c_{m_2}^{(r)}\bar{y}_r) \ne (c_1^{(r)}y_r^{(0)}, c_2^{(r)}y_r^{(0)}, \dots, c_{m_2}^{(r)}y_r^{(0)}).$$

So, we have:

$$D_r\bar{y}_r \le D_r y_r^{(0)}, D_r\bar{y}_r \ne D_r y_r^{(0)}. \tag{8}$$

Let $z^* = (x_0, y_1, \dots, y_{r-1}, \bar{y}_r, y_{r+1}, \dots, y_k)$. Obviously, $z^* \in Z$. we get:

$$Hz^* = \begin{pmatrix} 0 & D_1 & 0 & \cdots & 0 \\ 0 & 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D_k \\ -I & 0 & 0 & \cdots & 0 \\ e^t & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_1 \\ \vdots \\ \bar{y}_r \\ \vdots \\ y_k \end{pmatrix} = \begin{pmatrix} D_1 y_1 \\ \vdots \\ D_r \bar{y}_r \\ \vdots \\ D_k y_k \\ -x_0 \\ e^t x_0 \end{pmatrix},$$

and

$$Hz = \begin{pmatrix} 0 & D_1 & 0 & \cdots & 0 \\ 0 & 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D_k \\ -I & 0 & 0 & \cdots & 0 \\ e^t & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_1^{(0)} \\ y_2^{(0)} \\ \vdots \\ y_k^{(0)} \end{pmatrix} = \begin{pmatrix} D_1 y_1 \\ \vdots \\ D_r y_r \\ \vdots \\ D_k y_k \\ -x_0 \\ e^t x_0 \end{pmatrix}.$$

Due to (8), one has:

$$Hz^* \le Hz, Hz^* \ne Hz.$$

So, z^*dominates z with respect to the cone $K_2 = R_+^{m_2+n_1+1} \setminus \{0_{m_2+n_1+1}\}$, which contradicts $z \in E(H, Z, \le_{K_2})$.

$\Longrightarrow$ Let us show that $\Omega \subset E(H, Z, \le_{K_2})$.

Suppose that there is $z = (x_0, y_1^{(0)}, y_2^{(0)}, \dots, y_k^{(0)}) \in \Omega$ such that $z \notin E(H, Z, \le_{K_2})$. Therefore, there exists $z_1 = (x_1, y_1^{(1)}, \dots, y_k^{(1)})$ such that Hz_1 dominates Hz. This implies that $Hz_1 \le Hz$ and $Hz_1 \ne Hz$. Using the structure of the matrix H, one obtains:

$$Hz_1 = \begin{pmatrix} 0 & D_1 & 0 & \cdots & 0 \\ 0 & 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D_k \\ -I & 0 & 0 & \cdots & 0 \\ e^t & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1^{(1)} \\ y_2^{(1)} \\ \vdots \\ y_k^{(1)} \end{pmatrix} = \begin{pmatrix} D_1 y_1^{(1)} \\ \vdots \\ D_r y_r^{(1)} \\ \vdots \\ D_k y_k^{(1)} \\ -x_1 \\ e^t x_1 \end{pmatrix} \le \begin{pmatrix} 0 & D_1 & 0 & \cdots & 0 \\ 0 & 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D_k \\ -I & 0 & 0 & \cdots & 0 \\ e^t & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_1^{(0)} \\ y_2^{(0)} \\ \vdots \\ y_k^{(0)} \end{pmatrix} = \begin{pmatrix} D_1 y_1^{(0)} \\ \vdots \\ D_r y_r^{(0)} \\ \vdots \\ D_k y_k^{(0)} \\ -x_0 \\ e^t x_0 \end{pmatrix} = Hz,$$

and

$$Hz_1 = \begin{pmatrix} D_1 y_1^{(1)} \\ \vdots \\ D_r y_r^{(1)} \\ \vdots \\ D_k y_k^{(1)} \\ -x_1 \\ e^t x_1 \end{pmatrix} \ne \begin{pmatrix} D_1 y_1^{(0)} \\ \vdots \\ D_r y_r^{(0)} \\ \vdots \\ D_k y_k^{(0)} \\ -x_0 \\ e^t x_0 \end{pmatrix} = Hz.$$

This implies that $x_1 - x_0 \ge 0$ and $e^t(x_1 - x_0) \le 0$, which means that $x_1 = x_0$ and so $e^t x_1 = e^t x_0$. Then, we obtain

$$\begin{pmatrix} D_1 y_1^{(1)} \\ \vdots \\ D_r y_r^{(1)} \\ \vdots \\ D_k y_k^{(1)} \end{pmatrix} \le \begin{pmatrix} D_1 y_1^{(0)} \\ \vdots \\ D_r y_r^{(0)} \\ \vdots \\ D_k y_k^{(0)} \end{pmatrix}, \begin{pmatrix} D_1 y_1^{(1)} \\ \vdots \\ D_r y_r^{(1)} \\ \vdots \\ D_k y_k^{(1)} \end{pmatrix} \ne \begin{pmatrix} D_1 y_1^{(0)} \\ \vdots \\ D_r y_r^{(0)} \\ \vdots \\ D_k y_k^{(0)} \end{pmatrix}.$$

Thus, there exists $r \in \{1,2,\dots,k\}$ such that $D_r y_r^{(1)} < D_r y_r^{(0)}$. This implies that z_1 dominates z, which contradicts $y_i^{(0)} \in R_i(x_0)$, for $i = 1,2,\dots,k$. This completes the proof. ■

From this theorem, one can deduce that solving the MMLBPP is equivalent to solving the following problem:

$$min_{x,y_1,\dots,y_k} F(x,y_1,\dots,y_k) = \left(C_1(x,y_1,\dots,y_k), C_2(x,y_1,\dots,y_k),\dots, C_{m_1}(x,y_1,\dots,y_k)\right), \quad (9)$$

$$\text{s.t.} \quad (x,y_1,\dots,y_k) \in E(H,Z,\leq_{K_2}).$$

We denote by S the solution set of MMLBPP. We also have the following corollary:

Corollary 4.1. $S = E\left(F, E\left(H,Z,\leq_{K_2}\right),\leq_{K_1}\right)$ where $K_1 = R_+^{m_1} \setminus \{0_{m_1}\}$ and $K_2 = R_+^{m_2+n_1+1} \setminus \{0_{m_2+n_1+1}\}$.

5. A New Approach to Solving an MMLBPP

In this section, we propose an algorithm to generate a representative subset of $E\left(H,Z,\leq_{K_2}\right)$. Then, one can compute the image of the obtained subset by the leader objective functions and select elements that lead to the non-dominated points for the leader.

Consider the MMLBPP given by (3).

The Algorithm

Step 1: Construct the following multi-objective linear programming problem:

$$min_{x,y_1,\dots,y_k} H(x,y_1,\dots,y_k) = \begin{pmatrix} 0 & D_1 & 0 & \cdots & 0 \\ 0 & 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D_k \\ -I & 0 & 0 & \cdots & 0 \\ e^t & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x \\ y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix}, \quad \text{(LMPP1)}$$

$$\text{s.t.} \quad A_1 x \leq b_1,$$

$$A_2^{(i)} x + A_3^{(i)} y_i \leq b_2^{(i)}, \quad i = 1,2,\dots,k,$$
$$x \geq 0,\ y_i \geq 0, \quad i = 1,2,\dots,k.$$

Step 2: Compute a representative subset (called S) of the efficient solutions set of LMPP1.

- For instance, the approaches developed by Mattson et al. (2004), Messac & Mattson (2002) and Sayin (2003) can be exploited.

Step 3: Compute the image set of S by $F = (Y = F(S))$.

Step 4: Find the non-dominated points of Y (called Y_{eff}) with respect to F.

Step 5: Find the Pareto-optimal points set X_E corresponding to Y_{eff}.

- The Pareto-filter approaches presented by Mattson et al. (2004) can be used in Step 4 and Step 5.

Step 6: X_E is a representative subset of the efficient set of MMLBPP.

Stop.

5.1. Example

Consider the following MMLBP problem.

$$max_x\ (x + 2y_1 + 3y_2\ , y_3 - y_4), \quad (10)$$

$$s.t. \begin{cases} x \leq 5, \\ max_{y_1,y_2} (y_1 + y_2, y_2), \\ \quad s.t. \begin{cases} y_1 + y_2 \leq 30, \\ 10 \leq y_1 \leq 50, \\ 10 \leq y_2 \leq 40, \end{cases} \\ max_{y_3,y_4} (y_3 + y_4, y_3), \\ \quad s.t. \begin{cases} x + y_3 + y_4 \leq 40, \\ 10 \leq y_3 \leq 40, \\ 5 \leq y_4 \leq 30. \end{cases} \end{cases}$$

We construct the following multi-objective linear programming problem:

$$max_{x,y_1,\dots,y_4} H(x,y_1,y_2,y_3,y_4) \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix},$$

$$s.t. \begin{cases} y_1 + y_2 \leq 30, \\ x + y_3 + y_4 \leq 40, \\ x \leq 5, \\ 10 \leq y_1 \leq 50, \\ 10 \leq y_2, y_3 \leq 40, \\ 5 \leq y_4 \leq 30. \end{cases}$$

This is equivalent to the following problem:

$$min_{x,y_1,\dots,y_4} (-y_1 - y_2, -y_2,\ -y_3 - y_4,\ -y_3\ , x, -x), \quad (11)$$

$$s.t. \begin{cases} y_1 + y_2 \leq 30, \\ x + y_3 + y_4 \leq 40, \\ x \leq 5, \\ 10 \leq y_1 \leq 50, \\ 10 \leq y_2, y_3 \leq 40, \\ 5 \leq y_4 \leq 30. \end{cases}$$

Next, by applying the approaches presented by Mattson et al. (2004) for computing the efficient set, we obtain a representative subset of the efficient set of the problem (11). Then, go to step 3 and continue by the remaining steps of the algorithm. Tables 1 and 2 present non-dominated points and Pareto-optimal points for (10), respectively. The solution obtained with the approach presented by Bkay (2009) (this approach obtains only one solution) is the point $(5, 10, 20, 30, 5)$ with the values of leader objective functions equal to $F^* = (85, 25)$.

Table 1.
Non-dominated points

F_1	98.2640	95.1732	91.0149	87.9957	86.8587	
F_2	10.8450	14.8268	18.9851	22.0043	23.0814	
F_1	64.6149	77.1365	64.6737	79.9789	88.1846	80.1113
F_2	25	24.7103	24.7556	22.4359	21.6713	22.3556

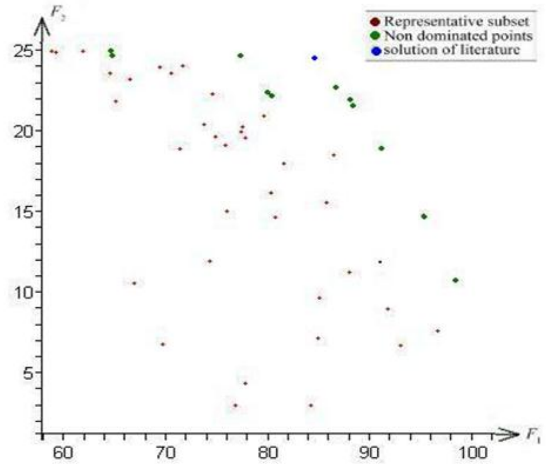

Fig. 1. Representation of non dominated points

Table 2
Pareto-optimal points

x	19.1549	15.1732	11.0149	7.9957	6.8567	
y_1	10.8910	10.0000	10.0000	10.0000	10	
y_2	19.1090	20.0000	20.0000	20.0000	20	
y_3	15.8450	19.8268	23.9851	27.0043	28.1124	
y_4	5.0000	5.0000	5.0000	5.0000	5.0309	
x	5.0000	5.0000	5.0000	7.5641	8.3237	7.6444
y_1	12.4463	14.5272	12.1397	17.5852	10.1391	17.5332
y_2	11.5741	14.3607	11.7981	12.4148	19.8609	12.4668
y_3	30.0000	29.8551	29.8151	27.4359	26.6738	27.3556
y_4	5.0000	5.1449	5.0596	5.0000	5.0025	5.0000

6. Conclusions

We have established a relation between the set of feasible solutions to an MMLBPP and the set of all efficient points of an artificial multi-objective linear programming problem. Generally, the purpose of solving a multi-objective programming problem is to obtain the representative set of efficient solutions. The approach presented for solving MMLBPP unlike the approaches presented by Ansari & Zhiani Rezai (2011) and Bkay (2009) that only obtain one solution, uses a Pareto-filter scheme and obtains an approximate discrete representation of the efficient set. Further studies can address practical applications and extend the similar techniques to solve multiobjective multilevel programming problems including more than two levels.

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