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On the topology of δ_e-open sets

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Abstract. The main purpose of this paper is to introduce and investigate a new class of open sets called δ_e-open sets. For this aim, first we define and study the notion of e-regular open set via e-closure operator. Then, we introduce the notion of δ_e-open set via e-regular open set. Several fundamental properties of the notion of δ_e-open set have been revealed. Also, we show that the family of δ_e-open sets is a topology strictly weaker than τ^δ and stronger than τ. In addition, we investigate relationships between the notion of δ_e-open set and other existing notions in topology such as open sets, regular open sets and δ-open sets. Furthermore, we give not only various properties and characterizations but also examples and counterexamples. Finally, some properties related to separation axioms are revealed.

Keywords: δ_e-open, δ_e-continuity, δ_e-Hausdorff space, δ_e-regular space, δ_e-normal space.

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1. Introduction

Undoubtedly, one of the basic concepts of general topology is different forms of the notion of open set. The discussion about different types of open sets is still a rich area to study in general topology. Some weak and strong forms of this concept such as regular open [15], δ-open set [16], θ-open set [16], e-open set [8], e^*-open set [7] and δ_ω-open set [3, 14] have been studied by many authors in the past years. Also, while the families of all δ-open and all θ-open subsets of a space X form a topology on X coarser than the old one, the family of all e-open subsets of a space X does not form a topology on

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X. In addition, many forms of continuity such as weakly e^*-continuity [10], contra $e^*\theta$-continuity [6], almost contra $e^*\theta$-continuity [5], weakly e^*-irresoluteness [12] and strongly e^*-irresoluteness [12] have been introduced through these new open set types defined. Apart from the above mentioned notions, many types of open set and continuity of functions have been studied in [1, 4, 11, 13] as well.

In this study, we introduce the concept of δ_e-open set, which lies between the concept of δ-open set and the concept of open set, by using the concept of e-open set and examine many fundamental properties of this newly defined concept. Moreover, we put forth that the family of all δ_e-open subsets of a topological space X forms a topology on X finer than τ. In addition, we define a new type of continuity via δ_e-open sets and investigate some of their characterizations. Furthermore, separation properties and some of their characterizations have been revealed.

In section 2, we provide the background information used throughout the paper. In section 3, we introduce the notion of e-regular open set and obtain some of its fundamental properties. Then, we define and investigate the notion of δ_e-open set via e-regular open set and examine its relationships with some of existing notions in the literature. In section 4, we introduce two new types of functions called the δ_e-open function and δ_e-closed function and obtain some characterizations of them. Also, we define a new type of continuity called δ_e-continuity. Moreover, we obtain its many characterizations and look into several fundamental properties. In section 5, we give the notions of δ_e-Hausdorff space, δ_e-regular space, and δ_e-normal space. In addition, we study not only their characterization, but also the relationships between them.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a topological space X, the closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively. The family of all closed (resp. open) sets of X is denoted by $C(X)$(resp. $O(X)$). In addition, the family of all open sets of X containing a point x of X is denoted by $O(X, x)$. Recall that a subset A of a space X is called regular open [15] (resp. regular closed [15]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). The family of all regular open subsets of X is denoted by $RO(X)$. The family of all regular open sets of X containing a point x of X is denoted by $RO(X, x)$. A subset A of a space X is called δ-open [16] if for each $x \in A$ there exists a regular open set V such that $x \in V \subseteq A$. A set A is said to be δ-closed if its complement is δ-open. The intersection of all regular closed sets of X containing A is called the δ-closure [16] of A and is denoted by δ-$cl(A)$. Dually, the union of all regular open sets of X contained in A is called the δ-interior [16] of A and is denoted by δ-$int(A)$. The θ-closure (resp. θ-interior) of a set A in a space X is defined by $\theta\text{-}cl(A) := \{x|(\forall U \in O(X, x))(cl(U) \cap A \neq \emptyset)\}$ (resp. $\theta\text{-}int(A) := \{x|(\exists U \in O(X, x))(cl(U) \subseteq A)\}$) [16]. A subset A is called θ-closed (resp. θ-open) if $\theta\text{-}cl(A) = A$ (resp. $\theta\text{-}int(A) = A$) [16]. The family of all δ-open subsets of X is denoted by $\delta O(X)$. The family of all δ-open sets of X containing a point x of X is denoted by $\delta O(X, x)$.

The ω-closure of a subset A of a topological space X is defined by $cl_\omega(A) := \{x|(\forall U \in O(X, x))(|U \cap A| > \aleph_0)\}$ [9]. A is called ω-closed if $cl_\omega(A) = A$ [9]. A is called ω-open if its complement is ω-closed [9]. The ω-interior of set A in topological space X is defined by $int_\omega(A) := \bigcup\{U|(U \subseteq A)(U \text{ is } \omega\text{-open})\}$. The θ_ω-closure of a set A in topological space X is defined by $\theta_\omega\text{-}cl(A) := \{x|(\forall U \in O(X, x))(cl_\omega(U) \cap A \neq \emptyset)\}$ [2]. A is called θ_ω-closed if θ_ω-$cl(A) = A$ [2]. A is called θ_ω-open if its complement is θ_ω-closed [2]. The θ_ω-interior of a set A in a space X is defined by $\theta_\omega\text{-}int(A) := \bigcup\{U|(U \subseteq A)(U \text{ is } \theta_\omega\text{-open})\}$.

The δ_ω-closure of a set A in topological space X is defined by $\delta_\omega\text{-}cl(A) := \{x|(\forall U \in O(X,x))(int(cl_\omega(U)) \cap A \neq \emptyset)\}$ [3]. A is called δ_ω-closed if δ_ω-$cl(A) = A$ [3]. A is called δ_ω-open if its complement is δ_ω-closed [3]. The δ_ω-interior of a set A in a space X is defined by $\delta_\omega\text{-}int(A) := \bigcup\{U|(U \subseteq A)(U \text{ is } \delta_\omega\text{-open})\}$.

We recall from [8] the followings:
A subset A of a space X is called e-open if $A \subseteq cl(\delta\text{-}int(A)) \cup int(\delta\text{-}cl(A))$. The complement of an e-open set is called e-closed. The intersection of all e-closed sets of X containing A is called the e-closure of A and is denoted by e-$cl(A)$. Dually, the union of all e-open sets of X contained in A is called the e-interior of A and is denoted by e-$int(A)$.

Lemma 2.1 [8] Let A and B be two subsets of a space X. Then the following statements hold:
a) $e\text{-}int(X \setminus A) = X \setminus e\text{-}cl(A)$,
b) $e\text{-}cl(X \setminus A) = X \setminus e\text{-}int(A)$,
c) $e\text{-}cl(A) \subseteq cl(A)$,
d) $int(A) \subseteq e\text{-}int(A)$,
e) $e\text{-}cl(A \cap B) \subseteq e\text{-}cl(A) \cap e\text{-}cl(B)$,
f) $e\text{-}int(A) \cup e\text{-}int(B) \subseteq e\text{-}int(A \cup B)$.

3. δ_e-open sets

Definition 3.1 A subset A of a space X is called e-regular open if $A = int(e\text{-}cl(A))$. The complement of an e-regular open set is called e-regular closed. The family of all e-regular open (resp. e-regular closed) subsets of X will be denoted by $eRO(X)$ (resp. $eRC(X)$).

Theorem 3.2 Let X be a topological space. Then $RO(X) \subseteq eRO(X) \subseteq O(X)$.

Proof. Let $A \in RO(X)$.
$$\left.\begin{array}{r} A \in RO(X) \Rightarrow A = int(cl(A)) \supseteq int(e\text{-}cl(A)) \\ A \in RO(X) \Rightarrow A = int(A) \subseteq int(e\text{-}cl(A)) \end{array}\right\} \Rightarrow A = int(e\text{-}cl(A))$$
$\Rightarrow A \in eRO(X)$.
Now, let $A \in eRO(X)$.
$A \in eRO(X) \Rightarrow A = int(e\text{-}cl(A)) \Rightarrow int(A) = int(e\text{-}cl(A)) = A \Rightarrow A \in \tau$. ∎

Remark 1 *The converses of the above inclusions need not be true as shown by the following examples.*

***Example* 3.3** Let $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, X, \{a,b\}, \{a,b,c\}\}$. Then simple calculations show that $RO(X) = \{\emptyset, X\}$ and $eRO(X) = \tau$. It is clear that the set $\{a,b,c\}$ is e-regular open but it is not regular open.

***Example* 3.4** Let $\mathbb{R}$ be the set of all real numbers with usual topology. Then the set $(0,1) \cup (1,2)$ is open but it is not e-regular open.

Theorem 3.5 Let X be a topological space and $A, B \subseteq X$. Then the following statements hold:
a) If A and B are two e-regular open sets, then $A \cap B$ is e-regular open.
b) A is e-regular closed if and only if $A = cl(e\text{-}int(A))$.

Proof. *a*) Let $A, B \in eRO(X)$.

$$\left.\begin{array}{l} A \in eRO(X) \Rightarrow A = int(e\text{-}cl(A)) \\ B \in eRO(X) \Rightarrow B = int(e\text{-}cl(B)) \end{array}\right\} \Rightarrow A \cap B = int(e\text{-}cl(A)) \cap int(e\text{-}cl(B))$$

$$\Rightarrow A \cap B = int(e\text{-}cl(A) \cap e\text{-}cl(B)) \supseteq int(e\text{-}cl(A \cap B)) \ldots (1)$$

$$\left.\begin{array}{r} A \cap B \subseteq e\text{-}cl(A \cap B) \Rightarrow int(A \cap B) \subseteq int(e\text{-}cl(A \cap B)) \\ A, B \in eRO(X) \subseteq O(X) \Rightarrow A \cap B \in O(X) \Rightarrow int(A \cap B) = A \cap B \end{array}\right\} \Rightarrow$$

$$\Rightarrow A \cap B \subseteq int(e\text{-}cl(A \cap B)) \ldots (2)$$

$$(1), (2) \Rightarrow A \cap B = int(e\text{-}cl(A \cap B)) \Rightarrow A \cap B \in eRO(X).$$

b) It is clear from Definition 3.1 and Lemma 2.1. ■

Remark 2 *The union of two e-regular open sets need not be e-regular open as shown by the following example.*

Example 3.6 Let $\mathbb{R}$ be the set of all real numbers with usual topology. Then the sets $(0, 1)$ and $(1, 2)$ are e-regular open but $(0, 1) \cup (1, 2)$ is not e-regular open.

Definition 3.7 Let X be a topological space and $A \subseteq X$. The δ_e-interior of A is the union of all e-regular open sets of X contained in A and is denoted by $\delta_e\text{-}int(A)$. Formally, $\delta_e\text{-}int(A) := \bigcup\{U | (U \subseteq A)(U \in eRO(X))\}$. If $A = \delta_e\text{-}int(A)$, then A is called a δ_e-open set. The complement of a δ_e-open set is called δ_e-closed. The family of all δ_e-open (resp. δ_e-closed) subsets of X will be denoted by $\delta_e O(X)$ (resp. $\delta_e C(X)$).

Theorem 3.8 Let X be a topological space and $A, B \subseteq X$. Then
a) $\delta_e\text{-}int(A) = \{x | (\exists U \in eRO(X, x))(U \subseteq A)\}$.
b) $\delta\text{-}int(A) \subseteq \delta_e\text{-}int(A) \subseteq A$.
c) $eRO(X) \subseteq \delta_e O(X)$.
d) $\delta O(X) \subseteq \delta_e O(X)$.
e) $\delta_e\text{-}int(A) \in O(X)$.
f) If A is open, then $A \subseteq int(e\text{-}cl(A))$.
g) $int(e\text{-}cl(int(e\text{-}cl(A)))) = int(e\text{-}cl(A))$.
h) $\delta_e\text{-}int(A) = \{x | (\exists U \in O(X, x))(int(e\text{-}cl(U)) \subseteq A)\}$.
i) If $A \subseteq B$, then $\delta_e\text{-}int(A) \subseteq \delta_e\text{-}int(B)$.
j) $\delta_e\text{-}int(A \cap B) = \delta_e\text{-}int(A) \cap \delta_e\text{-}int(B)$.
k) $\delta_e\text{-}int(A) \cup \delta_e\text{-}int(B) \subseteq \delta_e\text{-}int(A \cup B)$.
l) $\delta_e\text{-}int(\delta_e\text{-}int(A)) = \delta_e\text{-}int(A)$.
m) $\delta_e\text{-}int(A) \in \delta_e O(X)$.
n) $\delta_e\text{-}int(A) = \{x | (\exists U \in \delta_e O(X, x))(U \subseteq A)\}$.
o) $\delta_e\text{-}int(A) = max\{U | (U \subseteq A)(U \in \delta_e O(X))\}$, where the maximum is w.r.t. $\subseteq$.

Proof. Proofs are standard. Hence, they are omitted. ■

Theorem 3.9 Let X be a space and $A, B \subseteq X$. Then the followings hold:
a) $\emptyset, X \in \delta_e O(X)$.
b) If $A, B \in \delta_e O(X)$, then $A \cap B \in \delta_e O(X)$.
c) If $\mathcal{A} \subseteq \delta_e O(X)$, then $\bigcup \mathcal{A} \in \delta_e O(X)$.

Proof. *a*) $\delta_e\text{-}int(\emptyset) = \bigcup\{U | (U \subseteq \emptyset)(U \in eRO(X))\} = \bigcup\{\emptyset\} = \emptyset \Rightarrow \emptyset \in \delta_e O(X)$.
$\delta_e\text{-}int(X) = \bigcup\{U | (U \subseteq X)(U \in eRO(X))\} = \bigcup eRO(X) = X \Rightarrow X \in \delta_e O(X)$.
b) Let $A, B \in \delta_e O(X)$.

$$\left.\begin{array}{l} A \in \delta_e O(X) \Rightarrow A = \delta_e\text{-}int(A) \\ B \in \delta_e O(X) \Rightarrow B = \delta_e\text{-}int(B) \end{array}\right\} \Rightarrow A \cap B = \delta_e\text{-}int(A) \cap \delta_e\text{-}int(B)$$

$$\Rightarrow A \cap B = \delta_e\text{-}int(A) \cap \delta_e\text{-}int(B) = \delta_e\text{-}int(A \cap B) \Rightarrow A \cap B \in \delta_e O(X).$$

c) Let $\mathcal{A} \subseteq \delta_e O(X)$.

$\mathcal{A} \subseteq \delta_e O(X) \Rightarrow (\forall A \in \mathcal{A})(\delta_e\text{-}int(A) = A \subseteq \bigcup \mathcal{A})$

$$\left.\begin{array}{r} \Rightarrow (\forall A \in \mathcal{A})(A = \delta_e\text{-}int(A) \subseteq \delta_e\text{-}int\,(\bigcup \mathcal{A})) \Rightarrow \bigcup \mathcal{A} \subseteq \delta_e\text{-}int\,(\bigcup \mathcal{A}) \\ \mathcal{A} \subseteq \delta_e O(X) \Rightarrow \bigcup \mathcal{A} \subseteq X \Rightarrow \delta_e\text{-}int\,(\bigcup \mathcal{A}) \subseteq \bigcup \mathcal{A} \end{array}\right\} \Rightarrow$$

$\Rightarrow \bigcup \mathcal{A} = \delta_e\text{-}int\,(\bigcup \mathcal{A})$

$\Rightarrow \bigcup \mathcal{A} \in \delta_e O(X)$. ■

Corollary 3.10 The family of all δ_e-open sets in a space X is a topology on X.

Remark 3 *By Definitions 3.1 and 3.7 and Theorem 3.8, we have the following diagram for a subset of a topological space.*

$$\begin{array}{ccccccc} \theta\text{-}open \rightarrow & \delta\text{-}open & \rightarrow & \delta_e\text{-}open & \rightarrow open \rightarrow e\text{-}open \\ & \uparrow & & \uparrow & \\ & regular\ open & \rightarrow & e\text{-}regular\ open & \end{array}$$

The implications above are not reversible as shown in the following examples.

***Example* 3.11** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Simple calculations show that $\delta_e O(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then it is clear that the set $\{a, b, c\}$ is open but it is not δ_e-open.

***Example* 3.12** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}\}$. Simple calculations show that $\delta O(X) = \{\emptyset, X\}$ and $eRO(X) = \delta_e O(X) = \tau$. Then it is clear that the set $\{a, b, c\}$ is e-regular open and so δ_e-open but it is not δ-open.

***Example* 3.13** Let $\mathbb{R}$ be the set of all real numbers with usual topology. Then the set $(0, 1) \cup (1, 2)$ is δ_e-open but not e-regular open.

Definition 3.14 Let A be a subset of a space X. The δ_e-closure of A is the intersection of all e-regular closed sets of X containing A and is denoted by δ_e-$cl(A)$. Formally, $\delta_e\text{-}cl(A) := \bigcap \{F | (A \subseteq F)(F \in eRC(X))\}$.

Theorem 3.15 Let A be a subset of a space X. Then the following statements hold:
a) $\delta_e\text{-}cl(A) = \{x | (\forall U \in eRO(X, x))(U \cap A \neq \emptyset)\}$,
b) $\delta_e\text{-}cl(A) = \{x | (\forall U \in O(X, x))(int(e\text{-}cl(U)) \cap A \neq \emptyset)\}$,
c) $\delta_e\text{-}cl(A) = \{x | (\forall U \in \delta_e O(X, x))(U \cap A \neq \emptyset)\}$.

Proof. *a)* Let $A \subseteq X$.

$$\begin{array}{rl} x \notin \delta_e\text{-}cl(A) & \Leftrightarrow x \notin \bigcap \{F | (A \subseteq F)(F \in eRC(X))\} \\ & \Leftrightarrow (\exists F \in eRC(X))(A \subseteq F)(x \notin F) \\ & \Leftrightarrow (\setminus F \in eRO(X, x))(A \cap (\setminus F) \neq \emptyset) \\ & \Leftrightarrow x \notin \{x | (\forall U \in eRO(X, x))(U \cap A \neq \emptyset)\}. \end{array}$$

b) Let $x \notin \delta_e\text{-}cl(A)$.

$x \notin \delta_e\text{-}cl(A) \Rightarrow (\exists U \in eRO(X, x))(U \cap A = \emptyset)$

$$\left.\begin{array}{r} \Rightarrow (U \in eRO(X, x))(int(e\text{-}cl(U)) \cap A = \emptyset) \\ eRO(X) \subseteq O(X) \end{array}\right\} \Rightarrow$$

$\Rightarrow (U \in O(X, x))(int(e\text{-}cl(U)) \cap A = \emptyset)$

$\Rightarrow x \notin \{x | (\forall U \in O(X, x))(int(e\text{-}cl(U)) \cap A \neq \emptyset)\}$.

Now, let $x \notin B := \{x | (\forall U \in O(X, x))(int(e\text{-}cl(U)) \cap A \neq \emptyset)\}$.

$$\left.\begin{array}{r} x \notin B \Rightarrow (\exists U \in O(X, x))(int(e\text{-}cl(U)) \cap A = \emptyset) \\ W := int(e\text{-}cl(U)) \end{array}\right\} \Rightarrow$$

$\Rightarrow (W \in eRO(X,x))(W \cap A = \emptyset)$
$\Rightarrow x \notin \{x|(\forall U \in eRO(X,x))(U \cap A \neq \emptyset)\} \overset{(a)}{=} \delta_e\text{-}cl(A).$
c) Let $x \notin \delta_e\text{-}cl(A)$.

$$\left.\begin{array}{r} x \notin \delta_e\text{-}cl(A) \Rightarrow (\exists U \in eRO(X,x))(U \cap A = \emptyset) \\ eRO(X) \subseteq \delta_e O(X) \end{array}\right\} \Rightarrow$$

$\Rightarrow (U \in \delta_e O(X,x))(U \cap A = \emptyset)$
$\Rightarrow x \notin \{x|(\forall U \in \delta_e O(X,x))(U \cap A \neq \emptyset)\}.$
Now, let $x \notin \{x|(\forall U \in \delta_e O(X,x))(U \cap A \neq \emptyset)\}$.
$x \notin \{x|(\forall U \in \delta_e O(X,x))(U \cap A \neq \emptyset)\} \Rightarrow (\exists U \in \delta_e O(X,x))(U \cap A = \emptyset)$
$\Rightarrow (x \in U = \delta_e\text{-}int(U))(U \cap A = \emptyset)$
$\Rightarrow (\exists V \in eRO(X,x))(V \cap A \subseteq U \cap A = \emptyset)$
$\Rightarrow x \notin \delta_e\text{-}cl(A)$. ■

Corollary 3.16 The notions of closed set, δ_e-closed set and δ-closed set coincide in a regular topological space.

Theorem 3.17 Let A and B be two subsets of a space X. Then
a) If $A \in eRO(X)$, then $\delta_e\text{-}cl(A) = A$.
b) $\delta_e\text{-}cl(X \setminus A) = X \setminus \delta_e\text{-}int(A)$.
c) $\delta_e\text{-}int(X \setminus A) = X \setminus \delta_e\text{-}cl(A)$.
d) If $A \subseteq B$, then $\delta_e\text{-}cl(A) \subseteq \delta_e\text{-}cl(B)$.
e) $\delta_e\text{-}cl(A) \in \delta_e C(X)$.
f) $A \in \delta_e C(X)$ if and only if $A = \delta_e\text{-}cl(A)$.
g) $\delta_e\text{-}cl(A \cap B) \subseteq \delta_e\text{-}cl(A) \cap \delta_e\text{-}cl(B)$.
h) $\delta_e\text{-}cl(A \cup B) = \delta_e\text{-}cl(A) \cup \delta_e\text{-}cl(B)$.
i) $\delta_e\text{-}cl(\delta_e\text{-}cl(A)) = \delta_e\text{-}cl(A)$.

Proof. Proofs of the above results are standard, hence they are omitted. ■

Theorem 3.18 Let A and B be two subsets of a space X. Then the following properties hold:
a) If $A \in \delta_e O(X)$, then $A \cap \delta_e\text{-}cl(B) \subseteq \delta_e\text{-}cl(A \cap B)$.
b) If $A \in \delta_e C(X)$, then $\delta_e\text{-}int(A \cup B) \subseteq A \cup \delta_e\text{-}int(B)$.

Proof. *a*) Let $A \in \delta_e O(X)$ and $x \in A \cap \delta_e\text{-}cl(B)$.

$$\left.\begin{array}{r} x \in A \cap \delta_e\text{-}cl(B) \Rightarrow (x \in A)(\forall U \in \delta_e O(X,x))(U \cap B \neq \emptyset) \\ A \in \delta_e O(X) \end{array}\right\} \overset{\text{Theorem 3.9}}{\Rightarrow}$$

$\Rightarrow (\forall U \in \delta_e O(X,x))(U \cap A \in \delta_e O(X,x))((U \cap A) \cap B \neq \emptyset)$
$\Rightarrow (\forall U \in \delta_e O(X,x))(U \cap (A \cap B) \neq \emptyset)$
$\Rightarrow x \in \delta_e\text{-}cl(A \cap B)$.
b) This is obvious from (*a*) and Theorem 3.17(*b*). ■

Theorem 3.19 Let A be a subset of a space X. Then the followings hold:
a) If $A \in O(X)$, then $\delta_e\text{-}cl(A) = cl(A)$.
b) If $A \in C(X)$, then $\delta_e\text{-}int(A) = int(A)$.

Proof. *a*) For any subset A of a space X, we have always $cl(A) \subseteq \delta_e\text{-}cl(A) \ldots (1)$
Now, we will prove that $\delta_e\text{-}cl(A) \subseteq cl(A)$. Let $A \in O(X)$ and $x \notin cl(A)$.

$$\left.\begin{array}{r} x \notin cl(A) \Rightarrow (\exists U \in O(X,x))(U \cap A = \emptyset) \Rightarrow (\exists U \in O(X,x))(cl(U \cap A) = \emptyset) \\ A \in O(X) \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in O(X,x))(A \cap cl(U) \subseteq cl(U \cap A) = \emptyset)$
$\Rightarrow (\exists U \in O(X,x))(A \cap int(e\text{-}cl(U)) = \emptyset)$
$\Rightarrow x \notin \delta_e\text{-}cl(A)$

Then, we have $\delta_e\text{-}cl(A) \subseteq cl(A) \ldots (2)$
$(1),(2) \Rightarrow \delta_e\text{-}cl(A) = cl(A)$.
$b)$ It is obvious from (a) and Theorem 3.17(b). ■

Corollary 3.20 Let A be a subset of a space X. Then the following statements hold:
$a)$ $cl(\delta_e\text{-}int(A)) = \delta_e\text{-}cl(\delta_e\text{-}int(A))$.
$b)$ $int(\delta_e\text{-}cl(A)) = \delta_e\text{-}int(\delta_e\text{-}cl(A))$.
$c)$ $cl(\delta\text{-}int(A)) = \delta_e\text{-}cl(\delta\text{-}int(A))$.
$d)$ $int(\delta\text{-}cl(A)) = \delta_e\text{-}int(\delta\text{-}cl(A))$.
$e)$ $cl(\theta\text{-}int(A)) = \delta_e\text{-}cl(\theta\text{-}int(A))$.
$f)$ $int(\theta\text{-}cl(A)) = \delta_e\text{-}int(\theta\text{-}cl(A))$.
$g)$ $cl(\theta_\omega\text{-}int(A)) = \delta_e\text{-}cl(\theta_\omega\text{-}int(A))$.
$h)$ $int(\theta_\omega\text{-}cl(A)) = \delta_e\text{-}int(\theta_\omega\text{-}cl(A))$.
$i)$ $cl(\delta_\omega\text{-}int(A)) = \delta_e\text{-}cl(\delta_\omega\text{-}int(A))$.
$j)$ $int(\delta_\omega\text{-}cl(A)) = \delta_e\text{-}int(\delta_\omega\text{-}cl(A))$.

Proof. It is clear from Theorem 3.19. ■

4. Some fundamental properties

Definition 4.1 Let X and Y be two topological spaces. A function $f : X \to Y$ is said to be δ_e-open (resp. δ_e-closed) on X if $f[U]$ is δ_e-open (resp. δ_e-closed) in Y for every open (resp. closed) set U in X.

Theorem 4.2 Let X and Y be two topological spaces. If the function $f : X \to Y$ is bijective, then the notions of δ_e-open and δ_e-closed functions are equivalent.

Proof. Let f be a δ_e-open bijection and $A \in C(X)$.

$$\left.\begin{array}{r} A \in C(X) \Rightarrow \setminus A \in O(X) \\ f \text{ is a } \delta_e\text{-open bijection} \end{array}\right\} \Rightarrow \setminus f[A] = f[\setminus A] \in \delta_e O(Y) \Rightarrow f[A] \in \delta_e C(Y).$$

Now, let f be a δ_e-closed bijection and $A \in O(X)$.

$$\left.\begin{array}{r} A \in O(X) \Rightarrow \setminus A \in C(X) \\ f \text{ is a } \delta_e\text{-closed bijection} \end{array}\right\} \Rightarrow \setminus f[A] = f[\setminus A] \in \delta_e C(Y) \Rightarrow f[A] \in \delta_e O(Y).$$ ■

Theorem 4.3 Let X and Y be two topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:
$a)$ f is δ_e-open;
$b)$ $f[int(A)] \subseteq \delta_e\text{-}int(f[A])$ for every $A \subseteq X$;
$c)$ $f[B]$ is δ_e-open for every basic open set B in X;
$d)$ For each $x \in X$ and for every open set U in X containing x, there exists an open set V in Y containing $f(x)$ such that $int(e\text{-}cl(V)) \subseteq f[U]$.

Proof. $(a) \Rightarrow (b)$: Let $A \subseteq X$ and $y \in f[int(A)]$.
$y \in f[int(A)] \Rightarrow (\exists x \in int(A))(y = f(x))$

$$\left.\begin{array}{r} \Rightarrow (\exists U \in O(X,x))(U \subseteq A)(y = f(x)) \\ f \text{ is } \delta_e\text{-open} \end{array}\right\} \Rightarrow \ (f[U] \in \delta_e O(Y,y))(f[U] \subseteq f[A])$$

$\Rightarrow y \in \delta_e\text{-}int(f[A])$.
$(b) \Rightarrow (c)$: Let $\mathcal{B}$ be a base for topology on X and $B \in \mathcal{B}$.

$$\left.\begin{array}{r} B \in \mathcal{B} \Rightarrow B = int(B) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow f[B] = f[int(B)] \subseteq \delta_e\text{-}int(f[B]) \subseteq f[B]$$

$\Rightarrow \delta_e\text{-}int(f[B]) = f[B]$
$\Rightarrow f[B] \in \delta_e O(Y)$.

$(c) \Rightarrow (d)$: Let $x \in X$ and $U \in O(X,x)$.

$$\left.\begin{array}{r} U \in O(X,x) \\ \mathcal{B} \text{ is a base for topology on } X \end{array}\right\} \Rightarrow (\exists \mathcal{A} \subseteq \mathcal{B})(x \in U = \bigcup \mathcal{A})$$

$$\left.\begin{array}{r} \Rightarrow (\exists A \in \mathcal{A} \subseteq \mathcal{B})(x \in A \subseteq \bigcup \mathcal{A} = U) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (f[A] \in \delta_e O(Y, f(x)))(f[A] \subseteq f[U])$$

$$\left.\begin{array}{r} \Rightarrow (f[A] \in \delta_e O(Y, f(x)))(f[A] \subseteq f[U]) \\ V := f[A] \end{array}\right\} \Rightarrow (V \in \delta_e O(Y, f(x)))(V \subseteq f[U]).$$

$(d) \Rightarrow (a)$: Let $A \in O(X)$ and $y \in f[A]$.

$$\left.\begin{array}{r} y \in f[A] \Rightarrow (\exists x \in A)(y = f(x)) \\ A \in O(X) \end{array}\right\} \Rightarrow (\exists V \in \delta_e O(Y,y))(V \subseteq f[A])$$

$$\Rightarrow y \in \delta_e\text{-}int(f[A])$$

This means that $f[A] \subseteq \delta_e\text{-}int(f[A])$. On the other hand, we have always $\delta_e\text{-}int(f[A]) \subseteq f[A]$. Therefore, $f[A] = \delta_e\text{-}int(f[A])$. Thus, $f[A] \in \delta_e O(Y)$. ■

Theorem 4.4 Let X and Y be two topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:

a) f is δ_e-closed;

b) $\delta_e\text{-}cl(f[A]) \subseteq f[cl(A)]$ for each $A \subseteq X$.

Proof. $(a) \Rightarrow (b)$: Let $A \subseteq X$.

$$\left.\begin{array}{r} A \subseteq X \Rightarrow cl(A) \in C(X) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow f[cl(A)] \in \delta_e C(Y)$$

$$\left.\begin{array}{r} \Rightarrow \delta_e\text{-}cl(f[cl(A)]) = f[cl(A)] \\ A \subseteq X \Rightarrow \delta_e\text{-}cl(f[A]) \subseteq \delta_e\text{-}cl(f[cl(A)]) \end{array}\right\} \Rightarrow \delta_e\text{-}cl(f[A]) \subseteq f[cl(A)].$$

$(b) \Rightarrow (a)$: Let $A \in C(X)$.

$$\left.\begin{array}{r} A \in C(X) \Rightarrow A = cl(A) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \delta_e\text{-}cl(f[A]) \subseteq f[cl(A)] = f[A] \subseteq \delta_e\text{-}cl(f[A])$$

$$\Rightarrow f[A] = \delta_e\text{-}cl(f[A])$$

$$\Rightarrow f[A] \in \delta_e C(X).$$ ■

Definition 4.5 Let X and Y be two topological spaces. A function $f : X \to Y$ is said to be δ_e-continuous if $f^{-1}[O]$ is δ_e-open in X for every open set O in Y.

Theorem 4.6 Let (X,τ) and (Y,σ) be two topological spaces. A function $f : X \to Y$. Then the following statements are equivalent:

a) f is δ_e-continuous;

b) $f^{-1}[F]$ is δ_e-closed in X for each closed subset F of Y;

c) $f^{-1}[B]$ is δ_e-open in X for each (subbasic) basic open set B in Y;

d) For every $x \in X$ and every open set V of Y containing $f(x)$, there exists a δ_e-open set U containing x such that $f[U] \subseteq V$;

e) $f[\delta_e\text{-}cl(A)] \subseteq cl(f[A])$ for each subset A of X;

f) $\delta_e\text{-}cl(f^{-1}[B]) \subseteq f^{-1}[cl(B)]$ for each subset B of Y.

Proof. $(a) \Rightarrow (b)$: Let $F \in C(Y)$.

$$\left.\begin{array}{r} F \in C(Y) \Rightarrow \setminus F \in O(Y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \setminus f^{-1}[F] = f^{-1}[\setminus F] \in \delta_e O(X)$$

$$\Rightarrow f^{-1}(F) \in \delta_e C(X).$$

$(b) \Rightarrow (c)$: Let $\mathcal{B}$ be a base for σ and $B \in \mathcal{B}$.

$$\left.\begin{array}{r} B \in \mathcal{B} \\ \mathcal{B} \text{ is a base for } \sigma \Rightarrow \mathcal{B} \subseteq O(Y) \end{array}\right\} \Rightarrow B \in O(Y) \Rightarrow \setminus B \in C(Y)$$

$$\Rightarrow \setminus f^{-1}(B) = f^{-1}[\setminus B] \in \delta_e O(X)$$

$\Rightarrow f^{-1}[B] \in \delta_e C(X)$.

$(c) \Rightarrow (d)$: Let $x \in X$ and $V \in O(Y, f(x))$. Let $\mathcal{B}$ be a base for σ.

$$\left.\begin{array}{r} V \in O(Y, f(x)) \\ \mathcal{B} \text{ is a base for } \sigma \end{array}\right\} \Rightarrow (\exists \mathcal{A} \subseteq \mathcal{B} \subseteq O(Y))(\bigcup \mathcal{A} = V)$$

$$\left.\begin{array}{r} \Rightarrow (\exists A \in \mathcal{A} \subseteq \mathcal{B})(f(x) \in A \subseteq \bigcup \mathcal{A} = V) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (x \in f^{-1}[A] \in \delta_e O(X))(f^{-1}[A] \subseteq f^{-1}[V]) \\ U := f^{-1}[A] \end{array}\right\} \Rightarrow (U \in \delta_e O(X, x))(U \subseteq f^{-1}[V])$$

$\Rightarrow (U \in \delta_e O(X, x))(f[U] \subseteq V)$.

$(d) \Rightarrow (e)$: Let $A \subseteq X$ and $f(x) \notin cl(f[A])$.

$$\left.\begin{array}{r} f(x) \notin cl(f[A]) \Rightarrow (\exists V \in O(Y, f(x)))(V \cap f[A] = \emptyset) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in \delta_e O(X, x))(f[U \cap A] \subseteq f[U] \cap f[A] \subseteq V \cap f[A] = \emptyset)$
$\Rightarrow (\exists U \in \delta_e O(X, x))(U \cap A = \emptyset)$
$\Rightarrow x \notin \delta_e\text{-}cl(A)$
$\Rightarrow f(x) \notin f[\delta_e\text{-}cl(A)]$.

$(e) \Rightarrow (f)$: Let $B \subseteq Y$.

$$\left.\begin{array}{r} B \subseteq Y \Rightarrow f^{-1}[B] \subseteq X \\ \text{Hypothesis} \end{array}\right\} \Rightarrow f[\delta_e\text{-}cl(f^{-1}[B])] \subseteq cl(f[f^{-1}[B]]) \subseteq cl(B)$$

$\Rightarrow \delta_e\text{-}cl(f^{-1}[B]) \subseteq f^{-1}[B]$.

$(f) \Rightarrow (a)$: Let $A \in O(Y)$.

$$\left.\begin{array}{r} A \in O(Y) \Rightarrow \setminus A \in C(Y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \delta_e\text{-}cl(f^{-1}[\setminus A]) \subseteq f^{-1}[cl(\setminus A)] = f^{-1}[\setminus A]$$

$\Rightarrow \setminus \delta_e\text{-}int(f^{-1}[A]) = \delta_e\text{-}cl(\setminus f^{-1}[A]) = \delta_e\text{-}cl(f^{-1}[\setminus A]) \subseteq \setminus f^{-1}[A]$

$$\left.\begin{array}{r} \Rightarrow f^{-1}[A] \subseteq \delta_e\text{-}int(f^{-1}[A]) \\ A \subseteq Y \Rightarrow \delta_e\text{-}int(f^{-1}[A]) \subseteq f^{-1}[A] \end{array}\right\} \Rightarrow \delta_e\text{-}int(f^{-1}[A]) = f^{-1}[A]$$

$\Rightarrow f^{-1}[A] \in \delta_e O(X)$. ■

Theorem 4.7 Let X be a topological space and $Y = \prod\{Y_\alpha | \alpha \in I\}$ be a product space. A function $f : X \to Y$ is δ_e-continuous on X if and only if $p_\alpha \circ f$ is δ_e-continuous on X for every $\alpha \in I$.

Proof. $(\Rightarrow)$: Let $\alpha \in I$ and $U \in O(Y_\alpha)$.

$$\left.\begin{array}{r} U \in O(Y_\alpha) \\ \alpha \in I \Rightarrow p_\alpha \text{ is continuous} \end{array}\right\} \Rightarrow \left.\begin{array}{r} p_\alpha^{-1}[U] \in O(Y) \\ f \text{ is } \delta_e\text{-continuous} \end{array}\right\} \Rightarrow$$

$\Rightarrow f^{-1}[p_\alpha^{-1}[U]] = (p_\alpha \circ f)^{-1}[U] \in \delta_e O(X)$.

$(\Leftarrow)$: Let $p_\alpha \circ f$ be δ_e-continuous for all $\alpha \in I$ and $U \in O(Y)$.

$U \in O(Y) \Rightarrow (\forall \alpha \in I)(\exists U_\alpha \in Y_\alpha)\left(U = U_\alpha \times \prod_{\alpha \neq \beta} Y_\beta\right)$

$$\left.\begin{array}{r} \Rightarrow f^{-1}[U] = f^{-1}\left[U_\alpha \times \prod_{\alpha \neq \beta} Y_\beta\right] = f^{-1}[p_\alpha^{-1}[U_\alpha]] = (p_\alpha \circ f)^{-1}[U_\alpha] \\ p_\alpha \circ f \text{ is } \delta_e\text{-continuous} \end{array}\right\} \Rightarrow$$

$\Rightarrow f^{-1}[U] \in \delta_e O(X)$. ■

Corollary 4.8 Let X be a topological space and $Y = \prod\{Y_\alpha | \alpha \in I\}$ be a product space, and $f_\alpha : X \to Y_\alpha$ be a function for each $\alpha \in I$. Let $f : X \to Y$ be the function defined by $f(x) = \langle f_\alpha(x) \rangle$. Then f is δ_e-continuous on X if and only if f_α is δ_e-continuous on X for each $\alpha \in I$.

Proof. $(\Rightarrow)$: Let $\alpha \in I$.

$$\left.\begin{array}{r}\alpha \in I \Rightarrow p_\alpha \text{ is continuous} \\ f \text{ is } \delta_e\text{-continuous}\end{array}\right\} \Rightarrow f_\alpha = p_\alpha \circ f \text{ is } \delta_e\text{-continuous.}$$

$(\Leftarrow)$: Let $U \in O(Y)$.

$$\left.\begin{array}{r}U \in O(Y) \Rightarrow p_\alpha[U] \in O(Y_\alpha) \\ \alpha \in I \Rightarrow f_\alpha \text{ is } \delta_e\text{-continuous}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(p_\alpha^{-1} \circ f_\alpha)^{-1}[U] = f_\alpha^{-1}[p_\alpha[U]] \in \delta_e O(X) \\ \alpha \in I \Rightarrow p_\alpha \text{ is surjective}\end{array}\right\} \Rightarrow$$

$\Rightarrow f_\alpha^{-1}[p_\alpha[U]] = (p_\alpha^{-1} \circ f_\alpha)^{-1}[U] = f^{-1}[U] \in \delta_e O(X).$ ■

5. Separation properties and their characterizations

Definition 5.1 A topological space X is said to be

a) δ_e-Hausdorff if given any pair of distinct points x, y in X there exist disjoint δ_e-open sets U and V such that $x \in U$ and $y \in V$;

b) δ_e-regular if for each closed set F and each point $x \notin F$, there exist disjoint δ_e-open sets U and V such that $x \in U$ and $F \subseteq V$;

c) δ_e-normal if for every pair of disjoint closed sets E and F of X, there exist disjoint δ_e-open sets U and V such that $E \subseteq U$ and $F \subseteq V$.

Theorem 5.2 Let X be a topological space. Then the following statements are equivalent:

a) X is δ_e-Hausdorff;

b) For any distinct points x and y in X, there exists a δ_e-open set U containing x such that $y \notin \delta_e\text{-}cl(U)$;

c) $\bigcap\{\delta_e\text{-}cl(U) | U \in \delta_e O(X, x)\} = \{x\}$ for each $x \in X$.

Proof. $(a) \Rightarrow (b)$: Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}(x, y \in X)(x \neq y) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow (\exists U \in \delta_e O(X, x))(\exists V \in \delta_e O(X, y))(U \cap V = \emptyset)$$

$\overset{\text{Theorem 3.15(c)}}{\Rightarrow} y \notin \delta_e\text{-}cl(U).$

$(b) \Rightarrow (c)$: Let $x \in X$ ve $y \notin \{x\}$.

$$\left.\begin{array}{r}y \notin \{x\} \Rightarrow x \neq y \\ \text{Hypothesis}\end{array}\right\} \Rightarrow (\exists U \in \delta_e O(X, x))(y \notin \delta_e\text{-}cl(U))$$

$\Rightarrow y \notin \bigcap\{\delta_e\text{-}cl(U) | U \in \delta_e O(X, x)\}$

This means that

$$\bigcap\{\delta_e\text{-}cl(U) | U \in \delta_e O(X, x)\} \subseteq \{x\} \ldots (1)$$

$$\begin{array}{rl}y \notin \bigcap\{\delta_e\text{-}cl(U) | U \in \delta_e O(X, x)\} & \Rightarrow (\exists U \in \delta_e O(X, x))(y \notin \delta_e\text{-}cl(U)) \\ & \Rightarrow y \neq x \\ & \Rightarrow y \notin \{x\}\end{array}$$

This means that

$$\{x\} \subseteq \bigcap\{\delta_e\text{-}cl(U) | U \in \delta_e O(X, x)\} \ldots (2)$$

$(1), (2) \Rightarrow \bigcap\{\delta_e\text{-}cl(U) | U \in \delta_e O(X, x)\} = \{x\}.$

$(c) \Rightarrow (a)$: Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}(x, y \in X)(x \neq y) \Rightarrow y \notin \{x\} \\ \text{Hypothesis}\end{array}\right\} \Rightarrow y \notin \bigcap\{\delta_e\text{-}cl(U) | U \in \delta_e O(X, x)\}$$

$\Rightarrow (\exists U \in \delta_e O(X, x))(y \notin \delta_e\text{-}cl(U))$

$$\left.\begin{array}{r}\Rightarrow (\exists U \in \delta_e O(X,x))(y \in \setminus\delta_e\text{-}cl(U) = \delta_e\text{-}int(\setminus U) \in \delta_e O(X)) \\ V := \delta_e\text{-}int(\setminus U)\end{array}\right\} \Rightarrow$$
$\Rightarrow (U \in \delta_e O(X,x))(V \in \delta_e O(X,y))(U \cap V = \emptyset).$ ■

Theorem 5.3 Let X be a topological space. Then the following statements are equivalent:

a) X is δ_e-regular;

b) For each $x \in X$ and an open set U containing x, there exists a δ_e-open set V such that $x \in V \subseteq \delta_e\text{-}cl(V) \subseteq U$;

c) For each $x \in X$ and closed set F with $x \notin F$, there exists a δ_e-open set V containing x sucht that $F \cap \delta_e\text{-}cl(V) = \emptyset$.

Proof. $(a) \Rightarrow (b)$: Let $x \in X$ and $U \in O(X,x)$.
$$\left.\begin{array}{r}U \in O(X,x) \Rightarrow x \notin \setminus U \in C(X) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$
$\Rightarrow (\exists W \in \delta_e O(X,\setminus U))(\exists V \in \delta_e O(X,x))(W \cap V = \emptyset)$
$\Rightarrow (\exists W \in \delta_e O(X,\setminus U))(\exists V \in \delta_e O(X,x))(V \subseteq \setminus W)$
$\Rightarrow (\exists V \in \delta_e O(X,x))(\delta_e\text{-}cl(V) \subseteq \delta_e\text{-}cl(\setminus W) = \setminus W \subseteq U).$

$(b) \Rightarrow (c)$: Let $F \in C(X)$ and $x \notin F$.
$$\left.\begin{array}{r}x \notin F \in C(X) \Rightarrow x \in \setminus F \in O(X) \Rightarrow \setminus F \in O(X,x) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$
$\Rightarrow (\exists V \in \delta_e O(X,x))(\delta_e\text{-}cl(V) \subseteq \setminus F)$
$\Rightarrow (\exists V \in \delta_e O(X,x))(F \cap \delta_e\text{-}cl(V) = \emptyset).$

$(c) \Rightarrow (a)$: Let $F \in C(X)$ and $x \notin F$.
$$\left.\begin{array}{r}x \notin F \in C(X) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow (\exists V \in \delta_e O(X,x))(\delta_e\text{-}cl(V) \cap F = \emptyset)$$
$$\left.\begin{array}{r}\Rightarrow (\exists V \in \delta_e O(X,x))(F \subseteq \setminus\delta_e\text{-}cl(V)) \\ W := \setminus\delta_e\text{-}cl(V)\end{array}\right\} \Rightarrow$$
$\Rightarrow (\exists V \in \delta_e O(X,x))(\exists W \in \delta_e O(X,F))(V \cap W = \emptyset).$ ■

Theorem 5.4 Let X be a topological space. Then the following statements are equivalent:

a) X is δ_e-normal;

b) For each closed set F and an open set $U \supseteq F$, there exists a δ_e-open set V containing F such that $\delta_e\text{-}cl(V) \subseteq U$;

c) For each pair of distinct closed sets E and F, there exists a δ_e-open set V containing E such that $\delta_e\text{-}cl(V) \cap F = \emptyset$.

Proof. $(a) \Rightarrow (b)$: Let $F \in C(X)$ and $U \in O(X,F)$.
$$\left.\begin{array}{r}(U \in O(X,F))(F \in C(X)) \Rightarrow (F, \setminus U \in C(X))(F \cap (\setminus U) = \emptyset) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$
$\Rightarrow (\exists V \in \delta_e O(X,F))(\exists W \in \delta_e O(X,\setminus U))(W \cap V = \emptyset)$
$\Rightarrow (V \in \delta_e O(X,F))(W \in \delta_e O(X,\setminus U))(V \subseteq \setminus W)$
$\Rightarrow (V \in \delta_e O(X,F))(F \subseteq \delta_e\text{-}cl(V) \subseteq \delta_e\text{-}cl(\setminus W) = \setminus W \subseteq U)$
$\Rightarrow (V \in \delta_e O(X,F))(F \subseteq V \subseteq \delta_e\text{-}cl(V) \subseteq U).$

$(b) \Rightarrow (c)$: Let $E, F \in C(X)$ and $E \cap F = \emptyset$.
$$\left.\begin{array}{r}(E, F \in C(X))(E \cap F = \emptyset) \Rightarrow (E \in C(X))(\setminus F \in O(X,E)) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$
$\Rightarrow (\exists V \in \delta_e O(X,E))(E \subseteq \delta_e\text{-}cl(V) \subseteq \setminus F)$
$\Rightarrow (\exists V \in \delta_e O(X,E))(\delta_e\text{-}cl(V) \cap F = \emptyset).$

$(c) \Rightarrow (a)$: Let $E, F \in C(X)$ and $E \cap F = \emptyset$.

$$\left.\begin{array}{r}(E, F \in C(X))(E \cap F = \emptyset) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow (\exists V \in \delta_e O(X, E))(\delta_e\text{-}cl(V) \cap F = \emptyset)$$

$$\left.\begin{array}{r}\Rightarrow (\exists U \in \delta_e O(X, E))(F \subseteq \setminus \delta_e\text{-}cl(V) = \delta_e\text{-}int(\setminus V)) \\ W := \delta_e\text{-}int(\setminus V)\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists V \in \delta_e O(X, E))(\exists W \in \delta_e O(X, F))(V \cap W = \emptyset). \quad \blacksquare$$

Theorem 5.5 Let X be a T_1 space. Then the following statements hold:
a) If X is δ_e-regular, then X is δ_e-Hausdorff.
b) If X is δ_e-normal, then X is δ_e-regular.

Proof. *a*) Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}\left.\begin{array}{r}(x, y \in X)(x \neq y) \\ X \text{ is a } T_1 \text{ space}\end{array}\right\} \Rightarrow y \notin \{x\} \in C(X) \\ X \text{ is } \delta_e\text{-regular}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U \in \delta_e O(X, \{x\}) = \delta_e O(X, x))(\exists V \in \delta_e O(X, y))(U \cap V = \emptyset).$$

b) Let $x \notin F \in C(X)$.

$$\left.\begin{array}{r}\left.\begin{array}{r}x \notin F \in C(X) \\ X \text{ is a } T_1 \text{ space}\end{array}\right\} \Rightarrow (F, \{x\} \in C(X))(F \cap \{x\} = \emptyset) \\ X \text{ is } \delta_e\text{-normal}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U \in \delta_e O(X, F))(\exists V \in \delta_e O(X, \{x\}) = \delta_e O(X, x))(U \cap V = \emptyset). \quad \blacksquare$$

6. Conclusion

In this paper, we introduced a new type of open set called e-regular open. With the help of this new notion, we gave the notion of δ_e-open set and investigated some of its fundamental properties. Also, we proved that the family of all δ_e-open sets in a space (X, τ) is a topology strictly weaker than τ^δ and stronger than τ. We gave not only some relationships but also several examples. Moreover, we defined and studied some types of functions called δ_e-open, δ_e-closed and δ_e-continuity. Furthermore, we studied some separation properties defined in the scope of this present paper and investigated some of their characterizations. Finally, we believe that these results will play an important role for researchers who study in the area of general topology in future.

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